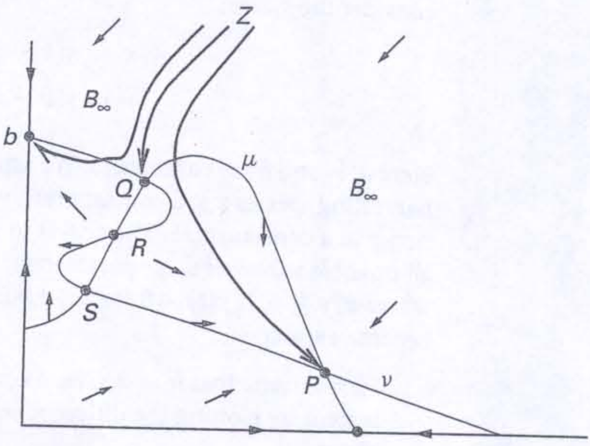

Figure 11.14 Note that solutions on either side of the point Z in the stable curve of Q have very different fates.

For example, this analysis tells us that, in Figure 11.14, only P and $(0, b)$ are asymptotically stable; all other equilibria are unstable. In particular, assuming that the equilibrium Q in Figure 11.14 is hyperbolic, then it must be a saddle because certain nearby solutions tend toward it, while others tend away. The point Z lies on one branch of the stable curve through Q. All points in the region denoted B_∞ to the left of Z tend to the equilibrium at $(0, b)$, while points to the right go to P. Thus as we move across the branch of the stable curve containing Z, the limiting behavior of solutions changes radically. Since solutions just to the right of Z tend to the equilibrium point P, it follows that the populations in this case tend to stabilize. On the other hand, just to the left of Z, solutions tend to an equilibrium point where $x = 0$. Thus in this case, one of the species becomes extinct. A small change in initial conditions has led to a dramatic change in the fate of populations. Ecologically, this small change could have been caused by the introduction of a new pesticide, the importation of additional members of one of the species, a forest fire, or the like. Mathematically, this event is a jump from the basin of P to that of $(0, b)$.

11.4 Exploration: Competition and Harvesting

In this exploration we will investigate the competitive species model where we allow either harvesting (emigration) or immigration of one of the species. We consider the system

$$x' = x(1 - ax - y)$$
$$y' = y(b - x - y) + h.$$

Here a, b, and h are parameters. We assume that $a, b > 0$. If $h < 0$, then we are harvesting species y at a constant rate, whereas if $h > 0$, we add to the population y at a constant rate. The goal is to understand this system completely for all possible values of these parameters. As usual, we only consider the regime where $x, y \geq 0$. If $y(t) < 0$ for any $t > 0$, then we consider this species to have become extinct.

1. First assume that $h = 0$. Give a complete synopsis of the behavior of this system by plotting the different behaviors you find in the a, b parameter plane.
2. Identify the points or curves in the ab–plane where bifurcations occur when $h = 0$ and describe them.
3. Now let $h < 0$. Describe the ab–parameter plane for various (fixed) h-values.
4. Repeat the previous exploration for $h > 0$.
5. Describe the full three-dimensional parameter space using pictures, flip books, 3D models, movies, or whatever you find most appropriate.

EXERCISES

1. For the SIRS model, prove that all solutions in the triangular region Δ tend to the equilibrium point $(\tau, 0)$ when the total population does not exceed the threshold level for the disease.

2. Sketch the phase plane for the following variant of the predator/prey system:

$$x' = x(1 - x) - xy$$
$$y' = y\left(1 - \frac{y}{x}\right).$$

3. A modification of the predator/prey equations is given by

$$x' = x(1 - x) - \frac{axy}{x + 1}$$
$$y' = y(1 - y)$$

where $a > 0$ is a parameter.