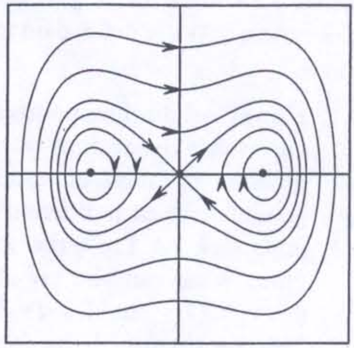

Figure 9.11 The phase portrait for $x' = y$, $y' = -x^3 + x$.

Proposition. *Suppose* (x_0, y_0) *is an equilibrium point for a planar Hamiltonian system. Then the eigenvalues of the linearized system are either* $\pm\lambda$ *or* $\pm i\lambda$ *where* $\lambda \in \mathbb{R}$. ∎

The proof of the proposition is straightforward (see Exercise 11).

9.5 Exploration: The Pendulum with Constant Forcing

Recall from Section 9.2 that the equations for a nonlinear pendulum are

$$\theta' = v$$
$$v' = -bv - \sin\theta.$$

Here θ gives the angular position of the pendulum (which we assume to be measured in the counterclockwise direction) and v is its angular velocity. The parameter $b > 0$ measures the damping.

Now we apply a constant torque to the pendulum in the counterclockwise direction. This amounts to adding a constant to the equation for v', so the system becomes

$$\theta' = v$$
$$v' = -bv - \sin\theta + k$$

where we assume that $k \geq 0$. Since θ is measured mod 2π, we may think of this system as being defined on the cylinder $S^1 \times \mathbb{R}$, where S^1 denotes the unit circle.

1. Find all equilibrium points for this system and determine their stability.
2. Determine the regions in the bk–parameter plane for which there are different numbers of equilibrium points. Describe the motion of the pendulum in each different case.
3. Suppose $k > 1$. Prove that there exists a periodic solution for this system. *Hint:* What can you say about the vector field in a strip of the form $0 < v_1 < (k - \sin\theta)/b < v_2$?
4. Describe the qualitative features of a Poincaré map defined on the line $\theta = 0$ for this system.
5. Prove that when $k > 1$ there is a unique periodic solution for this system. *Hint:* Recall the energy function
$$E(\theta, y) = \frac{1}{2}y^2 - \cos\theta + 1$$
and use the fact that the total change of E along any periodic solution must be 0.
6. Prove that there are parameter values for which a stable equilibrium and a periodic solution coexist.
7. Describe the bifurcation that must occur when the periodic solution ceases to exist.

EXERCISES

1. For each of the following systems, sketch the x- and y-nullclines and use this information to determine the nature of the phase portrait. You may assume that these systems are defined only for $x, y \geq 0$.

(a) $x' = x(y + 2x - 2)$, $y' = y(y - 1)$
(b) $x' = x(y + 2x - 2)$, $y' = y(y + x - 3)$
(c) $x' = x(2 - y - 2x)$, $y' = y(3 - 3y - x)$
(d) $x' = x(2 - y - 2x)$, $y' = y(3 - y - 4x)$
(e) $x' = x(2500 - x^2 - y^2)$, $y' = y(70 - y - x)$

2. Describe the phase portrait for
$$x' = x^2 - 1$$
$$y' = -xy + a(x^2 - 1)$$