Classification of Linear Systems

- Given a linear system $\dot{\mathbf{x}} = A\mathbf{x}$,
  the desired straight-line solutions ($\mathbf{x} = e^{\lambda t}\mathbf{v}$) exist
  if $\mathbf{v}$ is an **eigenvector** of A with **eigenvalue** λ, $A\mathbf{v} = \lambda\mathbf{v}$.
- Recall: **eigenvalues** of A is given by **characteristic equation**

$$\det(A - \lambda I)$$

  which has solutions

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\triangle}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\triangle}}{2}$$

  where $\tau = \mathrm{trace}(A) = a + d$ and $\triangle = \det(A) = ad - bc$.
- If $\lambda_1 \neq \lambda_2$ (typical situation),
  eigenvectors its $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linear independent**
  and span the entire plan.

General Solution for Initial Value Problem

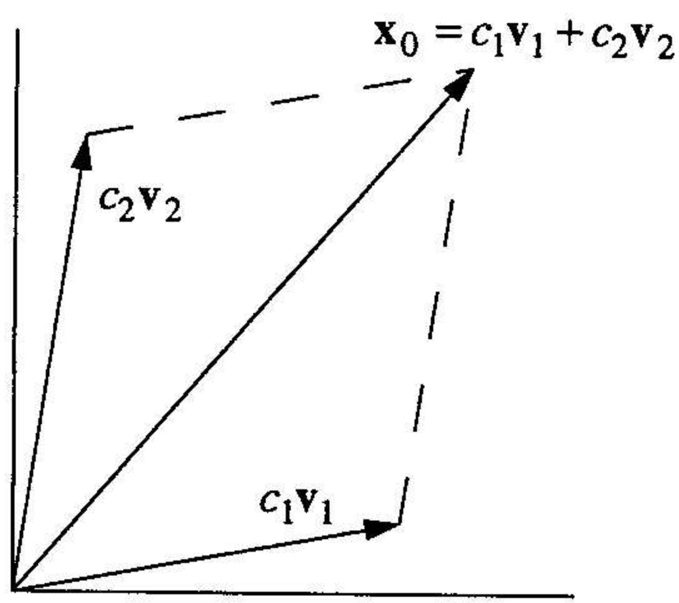
- Can write any **initial condition** as a **linear combination**

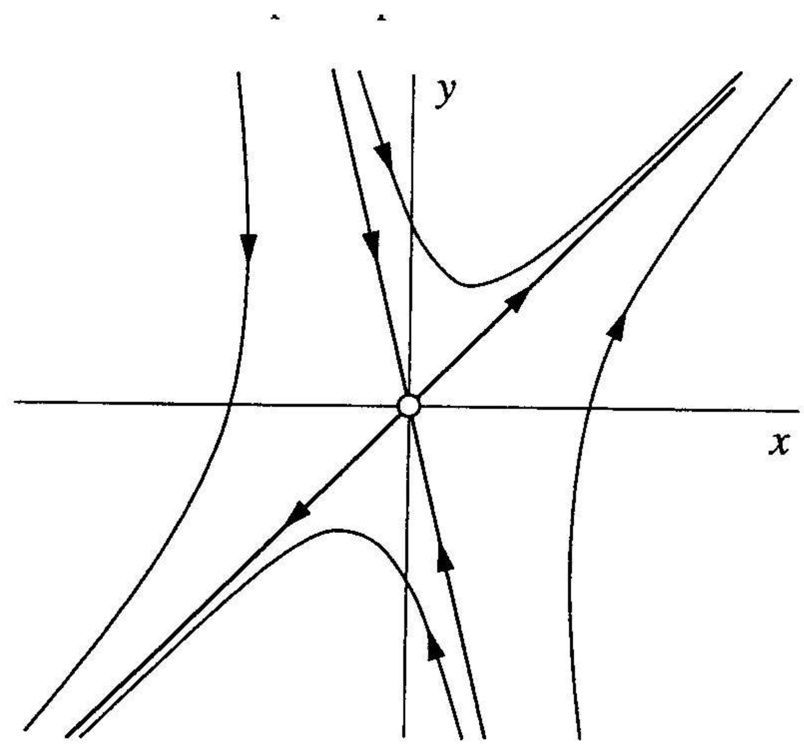
$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

- General solution is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2}\mathbf{v}_2$$

- Solve IVP $\dot{x} = x + y, \dot{y} = 4x - 2y$ with $(x_0, y_0) = (2, -3)$ and draw phase portrait.





Both Eigenvalues are Real and Different

- Have **opposite** signs: Fixed point is a **saddle**.
- Same sign (**negative**, **positive**): **nodes** (**stable**, **unstable**).

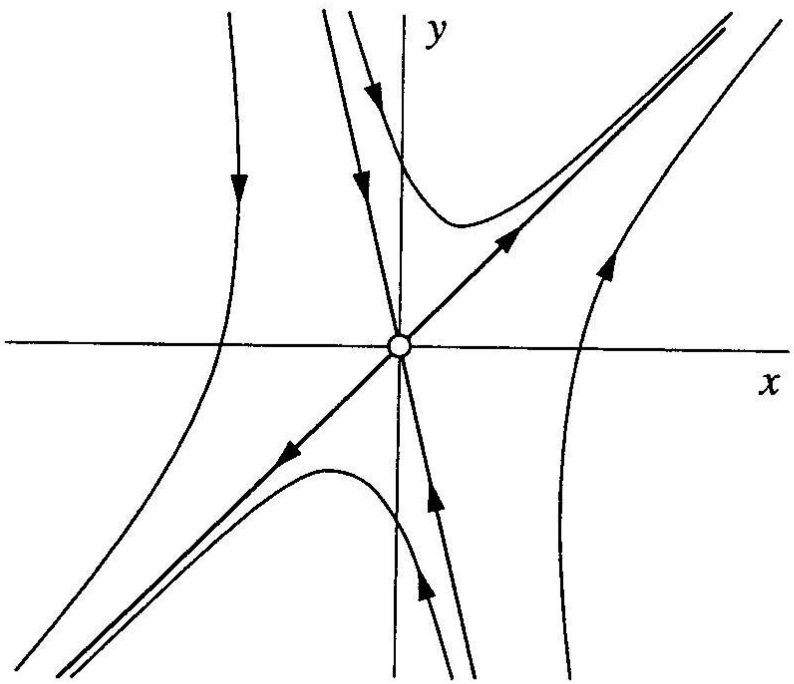


Figure 5.2.2

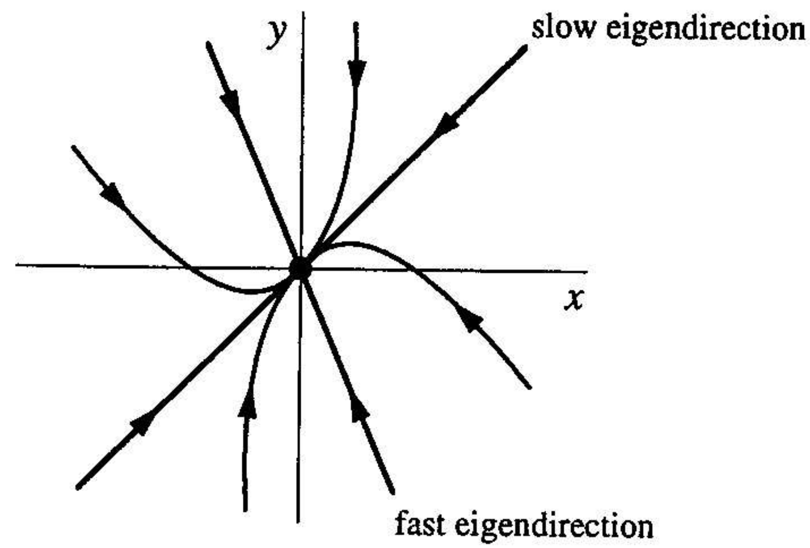


Figure 5.2.3

Eigenvalues are Complex Conjugates

- **Eigenvalues** are distinct

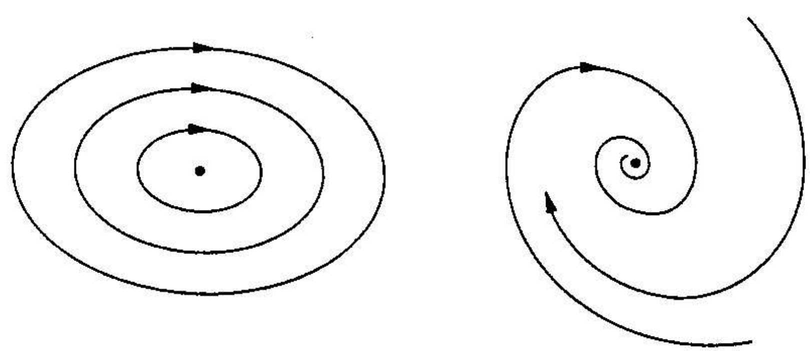
$$\lambda_{1,2} = \alpha \pm i\omega; \quad \alpha = \tau/2, \quad \omega = \tfrac{1}{2}\sqrt{4\Delta - \tau^2}$$

- General solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2}\mathbf{v}_2$$

  where c's and $\mathbf{v}$'s are **complex**.

- $\mathbf{x}(t)$ is a combination of $e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$.
  - **Decaying** oscillations if $\alpha = \text{Re}(\lambda) < 0$ (**stable spiral**)
  - **Growing** oscillations if $\alpha > 0$ (**unstable spiral**)



(a) center (b) spiral

Figure 5.2.4

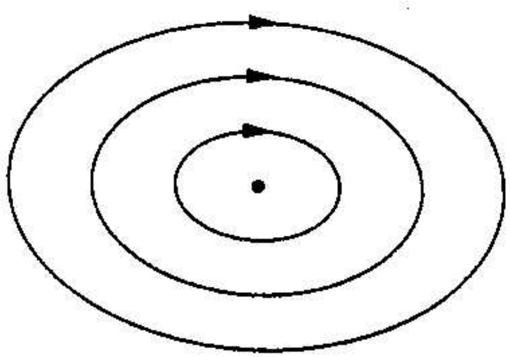
Eigenvalues are Complex Conjugates

- General solution is

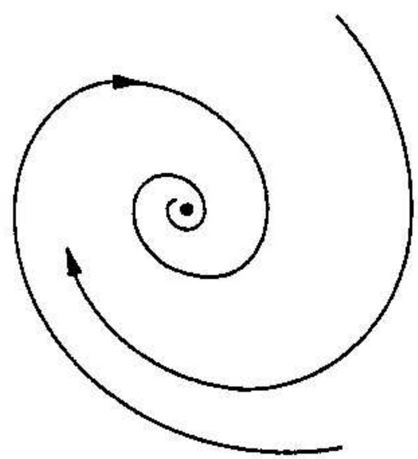
$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2}\mathbf{v}_2$$

where $\mathbf{x}(t)$ is a combination of $e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$.

- If **eigenvalues** are **purely imaginary** ($\alpha = 0$), all solutions are **periodic** with $T = 2\pi/\omega$
- Osicllations have fixed amplitude. Fixed point is a **center**.
- Easy to determine whether it is clockwise or counterclockwise.



(a) center



(b) spiral

Figure 5.2.4

Eigenvalues are Equal

- If 2 eigenvectors, every vector is eigenvector with eigenvalue λ.
- Since multiplication by A stretches every vector by λ,

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0.$$

  Trajectories are straight lines through $(0,0)$ (**star node**).
- If $\lambda = 0$, a plane of fixed points.

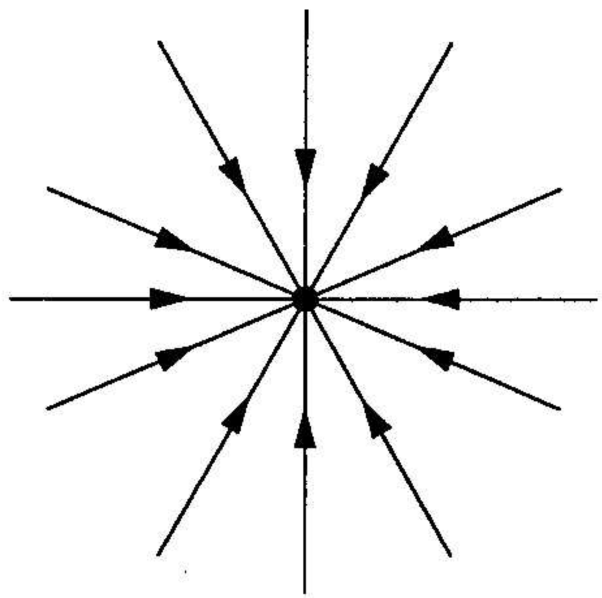


Figure 5.2.5

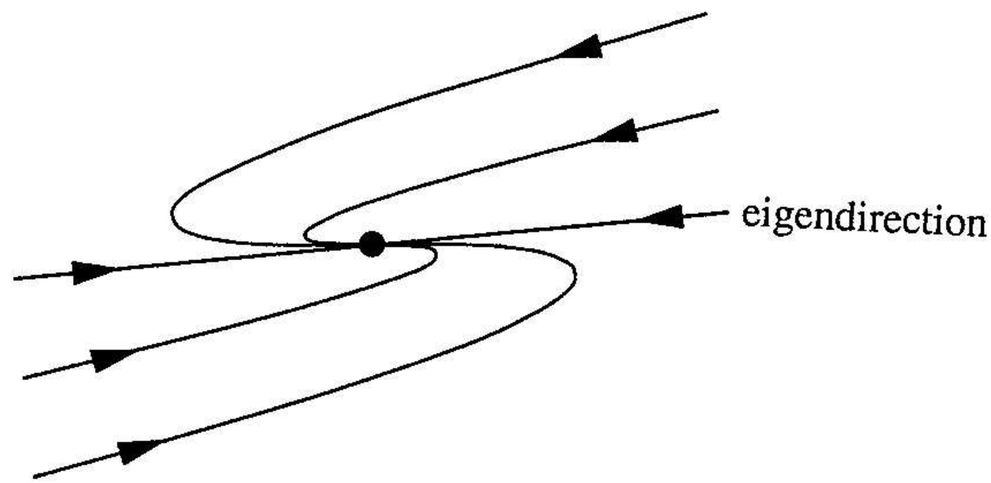


Figure 5.2.6

Eigenvalues are Equal

- If only 1 eigenvector, fixed point is **degenerate node**.
- Any matrix of the form

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

  has only a 1D eigenspace.
- As $t \to +\infty$, and $t \to -\infty$,
  all trajectories become **parallel** to only 1 eigendirection.

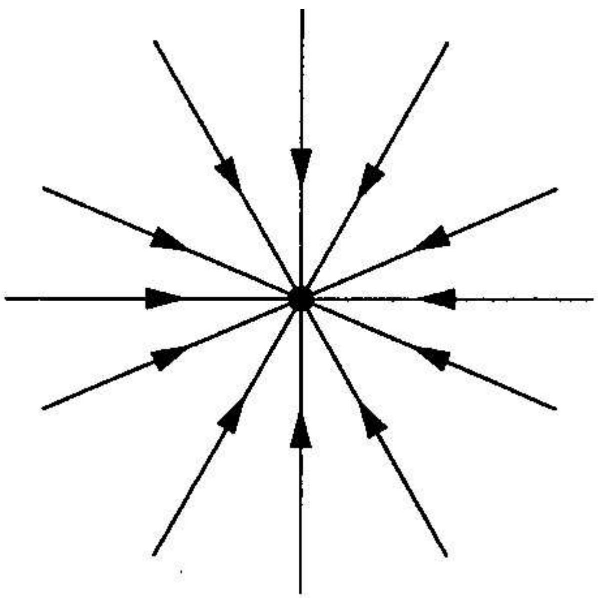


Figure 5.2.5

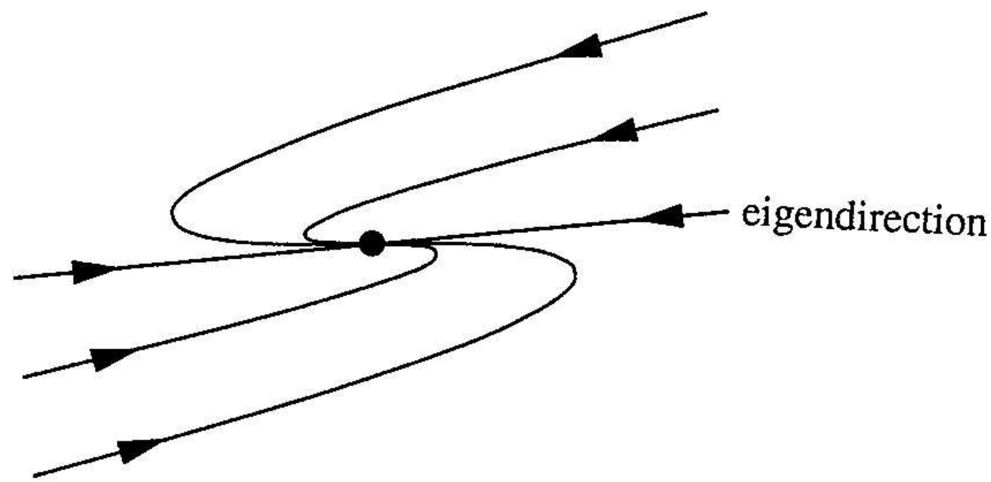


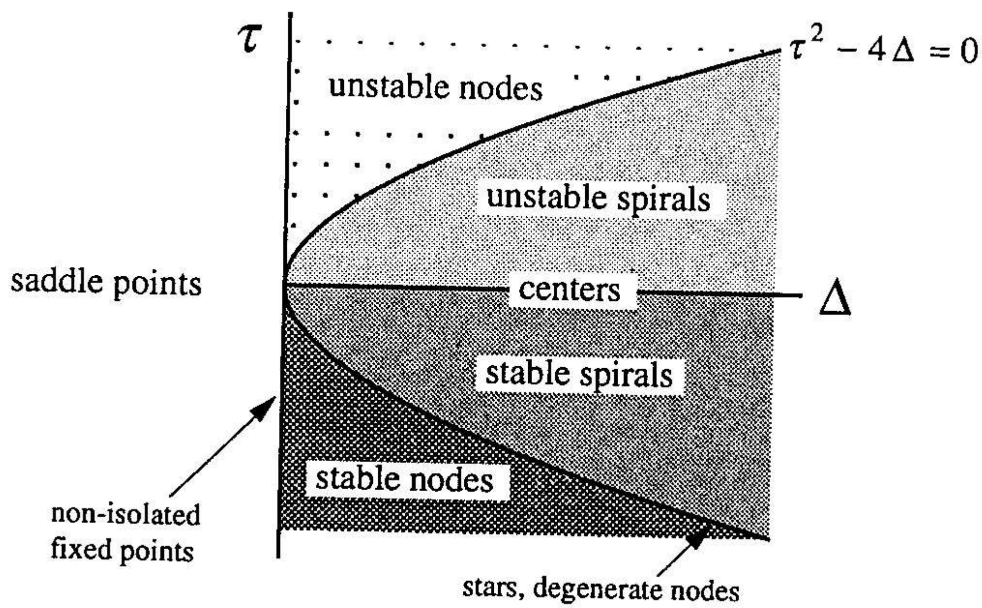
Figure 5.2.6

Classification of Fixed Points

- Recall:

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2.$$

- $\Delta < 0$: both are real, opposite signs. **Saddle**.
- $\Delta > 0$, $\tau^2 - 4\Delta > 0$: both are real, same sign. Nodes,
- $\Delta > 0$, $\tau^2 - 4\Delta < 0$: complex conjugates. Center and Spiral.
- $\tau^2 - 4\Delta = 0$, both are equal. Star Nodes. Degenerate Nodes.



Classification of Fixed Points

- Recall:

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\triangle}\right), \qquad \triangle = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2.$$

- $\triangle < 0$: real, opposite signs. **Saddle**.
- $\triangle > 0$, $\tau^2 - 4\triangle > 0$: real, same sign. Nodes. $\tau < 0$, $\tau > 0$.
- $\triangle > 0$, $\tau^2 - 4\triangle < 0$: complex conjugates. Center and Spiral.
- $\tau^2 - 4\triangle = 0$, both are equal. Star Nodes. Degenerate Nodes.