Two-Dimensional Linear System

- Simplest class of high-dimensional systems. Important for classification of **fixed points** of **nonlinear systems**.
- Two-dimensional linear system:

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

- Written compactly in **matrix form**

$$\dot{\mathbf{x}} = A\mathbf{x}; \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- **Linear**: if $\mathbf{x}_1, \mathbf{x}_2$ are solutions, so is $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.
- $\mathbf{x}^* = 0$ is always a **fixed point** for any A.
- Solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on (x, y) plane, called **phase plane**.

Vibrations of a Mass Hanging from a Linear Spring

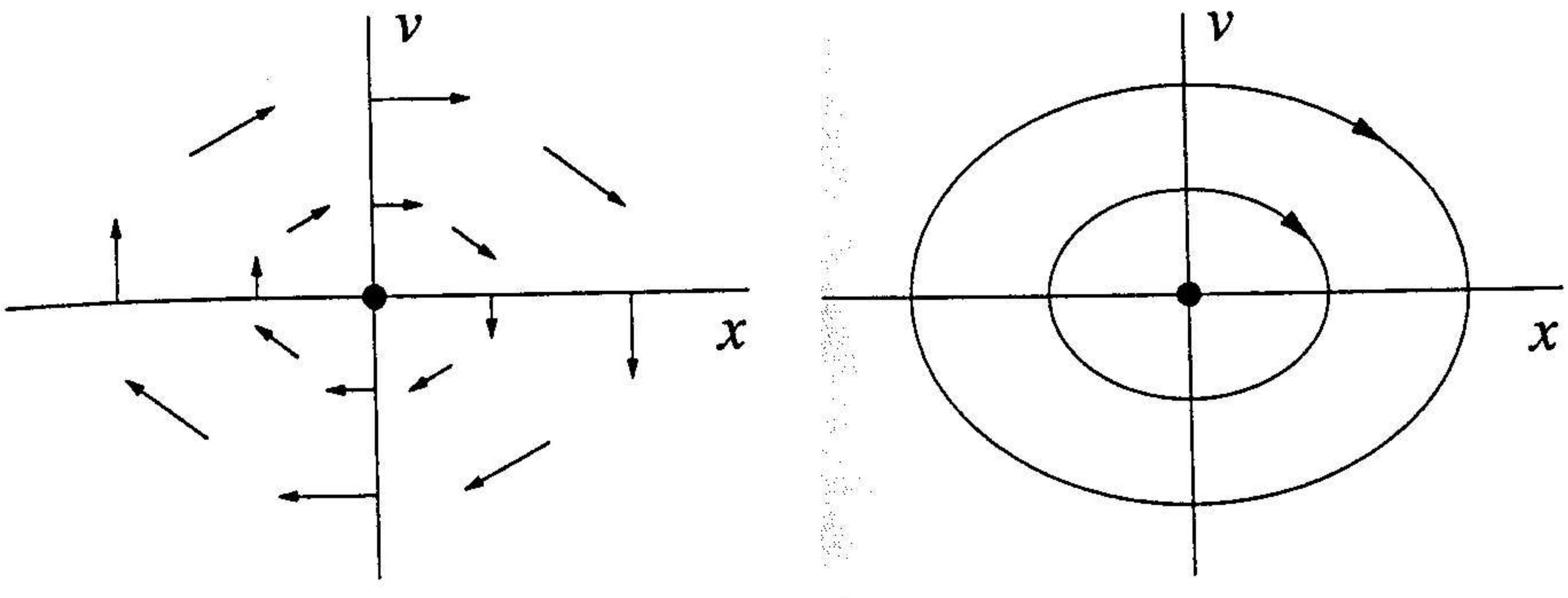
- Phase plan analysis of **Simple harmonic oscillator**

$$m\ddot{x} + kx = 0.$$

- Rewritten as

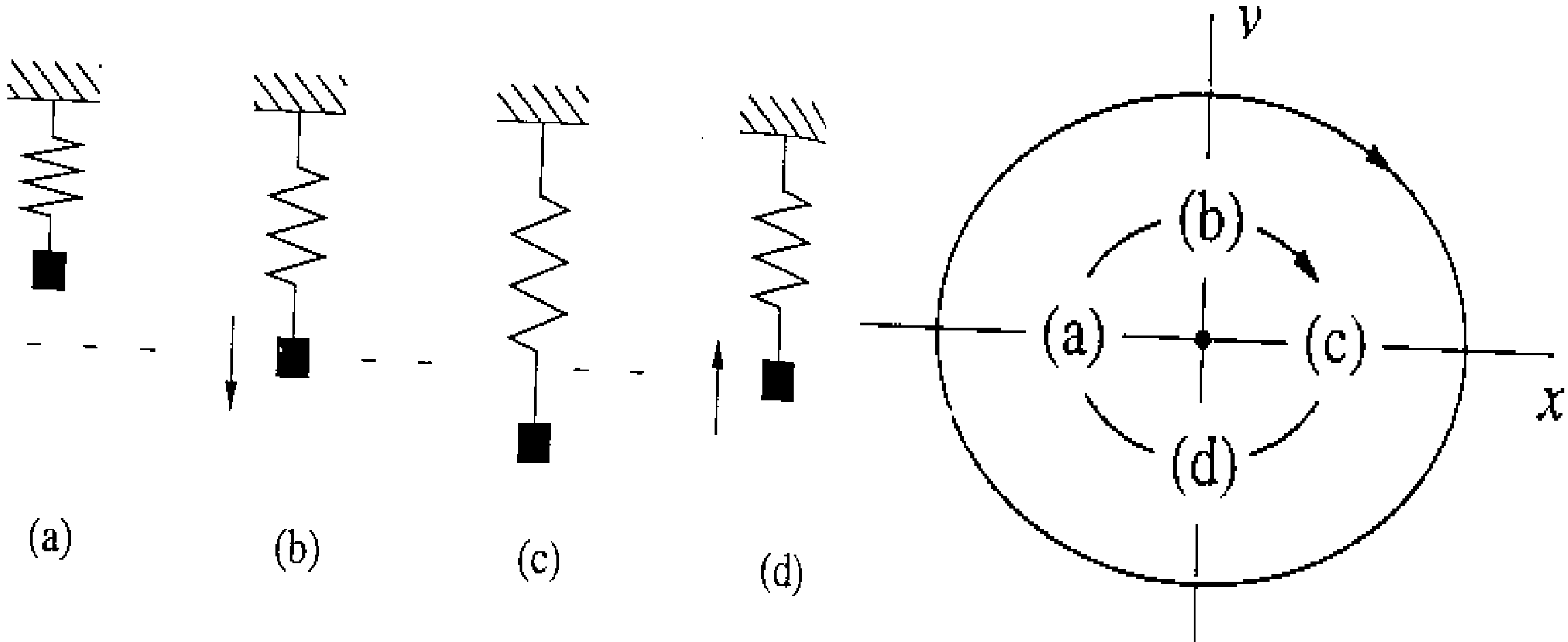
$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x = -\omega^2 x \end{aligned}$$

- **Vector Field**: assign a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at (x, v).



Fixed Point, Closed Orbits, and Phase Portrait

- **Fixed point** $(0,0)$: mass at rest at its equilibrium position.
- **Closed orbit**: periodic motion, oscillation of mass.
  - **(a)**: x most negative, $v = 0$; SP most compressed.
  - **(a) to (b)**: x increases, $v > 0$; M pushed back to equil.
  - **(b)**: M at $x = 0$, has largest $v > 0$, overshoots
  - **(b) to (c)** M eventually comes to rest at end of SP.
  - **(c) to (a)**: M gets pulled up, eventually completes cycle.

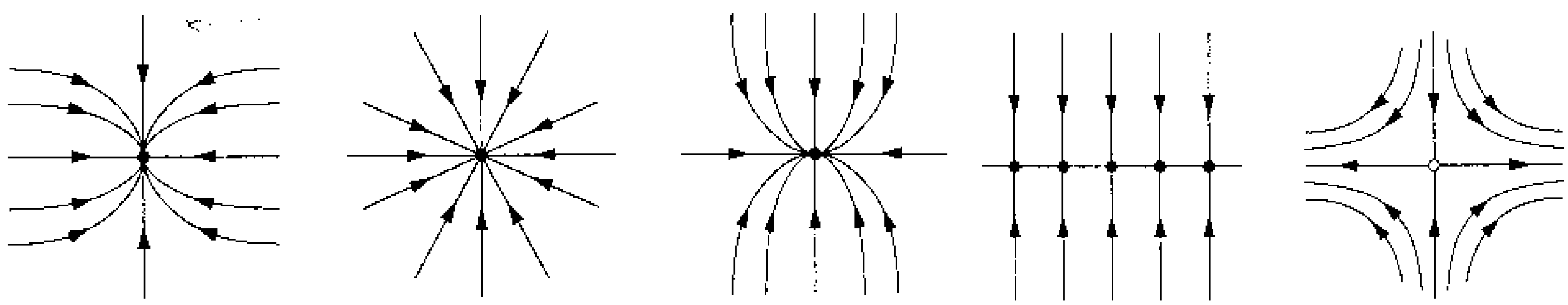


Different Cases of an Uncoupled Linear System

- Cosider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Solution: $x(t) = x_0 e^{at}$ and $y(t) = y_0 e^{-t}$.
  - (a) $a < -1$: $x(t)$ decays faster than $y(t)$; x^* is **stable node**.
  - (b) $a = -1$: **symmetrical node**.
  - (c) $-1 < a < 0$; $x(t)$ decays slower than $y(t)$; **stable node**.
  - (d) $a = 0$: $x(t) = x_0$; an **line of fixed points**.
  - (e): $a > 0$: $x(t)$ grows exponentially; x^* is **saddle point**. x-axis, **unstable manifold**; y-axis, **stable manifold** of x^*



(a) $a < -1$ (b) $a = -1$ (c) $-1 < a < 0$ (d) $a = 0$ (e) $a > 0$