Adaptive Control: CDS 270

Lecture 4

6. Invariance – like Theorems

Reading material:
[1]: Section 4.5
[2]: Section 8.3

For autonomous systems, LaSalle's invariance set theorems allow asymptotic stability conclusions to be drawn even when $\dot{V}$ is only negative semi-definite in a domain Ω. In that case, the system trajectory approaches the largest invariant set E, which is a subset of all points $x \in \Omega$ where $\dot{V}(x) = 0$. However the invariant set theorems are not applicable to nonautonomous systems. In the case of the latter, it may not even be clear how to define a set E, since V may explicitly depend on both t and x. Even when $V = V(x)$ does not explicitly depend on t the nonautonomous nature of the system dynamics precludes the use of the LaSalle's invariant set theorems.

Example 6.1
The closed-loop error dynamics of an adaptive control system for 1st order plant with one unknown parameter is

$$\dot{e} = -e + \theta\, w(t)$$
$$\dot{\theta} = -e\, w(t)$$

where e represents the tracking error and $w(t)$ is a bounded function of time t. Due to the presence of $w(t)$, the system dynamics is nonautonomous. Consider the Lyapunov function candidate

$$V(e,\theta) = e^2 + \theta^2$$

Its time derivative along the system trajectories is

$$\dot{V}(e,\theta) = 2e\dot{e} + 2\theta\dot{\theta} = 2e\left(-e + \theta\, w(t)\right) + 2\theta\left(-e\, w(t)\right) = -2e^2 \le 0$$

This implies that V is a decreasing function of time, and therefore, both $e(t)$ and $\theta(t)$ are bounded signals of time. But due to the nonautonomous nature of the system dynamics, the LaSalle's *invariance set theorems cannot be used* to conclude the convergence of $e(t)$ to the origin.

In general, if $\dot{V}(t,x) \le -W(x) \le 0$ then we may expect that the trajectory of the system approaches the set $\{W(x)=0\}$, as $t \to \infty$. Before we formulate main results, we state a lemma that is interesting in its own sake. The lemma is an important result about asymptotic properties of functions and their derivatives and it is known as the Barbalat's lemma.

Definition 6.1 (uniform continuity)
A function $f(t): R \to R$ is said to be *uniformly continuous* if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 \quad \forall |t_2 - t_1| \le \delta \Rightarrow |f(t_2) - f(t_1)| \le \varepsilon$$

Note that t_1 and t_2 play a symmetric role in the definition of the uniform continuity.

Lemma 6.1 (Barbalat)
Let $f(t): R \to R$ be differentiable and has a finite limit as $t \to \infty$. If $\dot{f}(t)$ is uniformly continuous then $\dot{f}(t) \to 0$ as $t \to \infty$, (see proof in [2], p. 323).

Lemma 6.2
If $\dot{f}(t)$ is bounded then $f(t)$ is uniformly continuous.

An immediate and practical corollary of Barbalat's lemma can now be stated.

Corollary 6.1
If $f(t): R \to R$ is twice differentiable, has a finite limit, and its 2nd derivative is bounded then $\dot{f}(t) \to 0$ as $t \to \infty$.

In general, the fact that derivative tends to zero does not imply that the function has a limit. Also, the converse is not true. In other words:

$$f(t) \to C \not\Leftrightarrow \dot{f}(t) \to 0$$

Example 6.2

- As $t \to \infty$, $f(t) = \sin(\ln t)$ does not have a limit, while $\dot{f}(t) = \dfrac{\cos(\ln t)}{t} \to 0$.

- As $t \to \infty$, $f(t) = e^{-t}\sin\left(e^{2t}\right) \to 0$, while $\dot{f}(t) = -e^{-t}\sin\left(e^{2t}\right) + e^{t}\sin\left(e^{2t}\right) \to \infty$.

Example 6.3
Consider an LTI system

$$\dot{x} = A\,x + B\,u$$

with a Hurwitz matrix A and a uniformly bounded in time input $u(t)$. These two facts imply that the state $x(t)$ is bounded. Thus, the state derivative $\dot{x}(t)$ is bounded. Let $y = C\,x$ represent the system output. Then $\dot{y} = C\,\dot{x}$ is bounded and, consequently the system output $y(t)$ is a *uniformly continuous* function of time. Moreover, if the input $u(t) = u_0$ is constant then the output $y(t)$ tends to a limit, as $t \to \infty$. The latter combined with the fact that y is uniformly continuous implies, (by Barbalat's Lemma), that the output time derivative $\dot{y}$ asymptotically approaches zero.

To apply Barbalat's lemma to the analysis of nonautonomous dynamic systems we state the following immediate corollary.

Corollary 6.2 (Lyapunov-like Lemma)
If a scalar function $V = V(t,x)$ is such that

- $V(t,x)$ is lower bounded
- $\dot{V}(t,x)$ is negative semi-definite along the trajectories of $\dot{x} = f(t,x)$
- $\dot{V}(t,x)$ is uniformly continuous in time

then $\dot{V}(t,x) \to 0$, as $t \to \infty$.

Notice that the first two assumptions imply that $V(t,x)$ tends to a limit. The latter coupled with the 3rd assumption proves (using Barbalat's lemma) the corollary.

Example 6.4
Consider again the closed-loop error dynamics of an adaptive control system from Example 6.1. Choosing $V(e,\theta) = e^2 + \theta^2$, it was shown that along the system trajectories: $\dot{V}(e,\theta) = -2e^2 \le 0$. The 2nd time derivative of V is

$$\ddot{V}(e,\theta) = -4\,e\,\dot{e} = -4\,e\left(-e + \theta\,w(t)\right)$$

Since $w(t)$ is bounded by hypothesis, and $e(t)$ and $\theta(t)$ were shown to be bounded, it is clear that $\ddot{V}$ is bounded.. Hence, $\dot{V}$ is uniformly continuous and by the Barbalat's lemma (or the Lyapunov-like lemma), $\dot{V} \to 0$ which in turn indicates that the tracking error $e(t)$ tends to zero, as $t \to \infty$.

7. Basic Concepts and Introduction to Adaptive Control

Reading material:
[1]: Chapter 8, Section 8.1
[2]: Section 1.2.6
[2]: Section 4.2, Example 4.10
[2]: Section 12.1

Introduction

Since the 1950's adaptive control has firmly remained in the mainstream of controls and dynamics research, and it has grown to become a well-formed scientific discipline. One of the reasons for the continuing popularity and rapid growth of adaptative control is its clearly defined goal – to control dynamical systems with unknown parameters.

Research in adaptive control started in connection with the design of autopilots for high-performance aircraft. But interest in the subject has soon diminished due to the lack of insights and the crash of a test flight, (NASA X-15 program). The last decade has witnessed the development of a coherent theory for adaptive control, which has lead to many practical applications in the areas such as aerospace, robotics, chemical process control, ship steering, bioengineering, and many others.

The basic idea in adaptive control is to estimate the uncertain plant and / or controller parameters on-line based on the measured system signals and use the estimated parameters in control input computation. An adaptive controller can thus be regarded as an inherently nonlinear dynamic system with on-line parameter estimation.

Generally speaking, *the basic objective* of adaptive control is to maintain consistent performance of a system in the presence of uncertainty or unknown variation in plant parameters.

There are two main approaches for constructing adaptive controllers:

- Model reference adaptive control (MRAC) method
- Self-tuning control (STC) method

Schematic representation of an MRAC system is given in Figure 7.1.

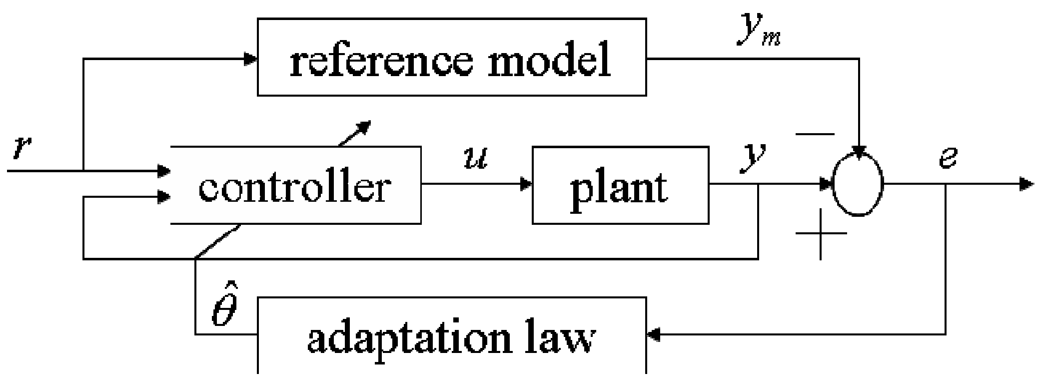

Figure 7.1: Model Reference Adaptive Control System

The MRAC system is composed of *four parts*:

- Plant of a known structure but with unknown parameters
- Reference model for specification of the desired system output
- Feedback / feedforward control law with adjustable gains, (controller)
- Parameter / gain adaptation law

Schematic representation of an STC system is given in Figure 7.2.

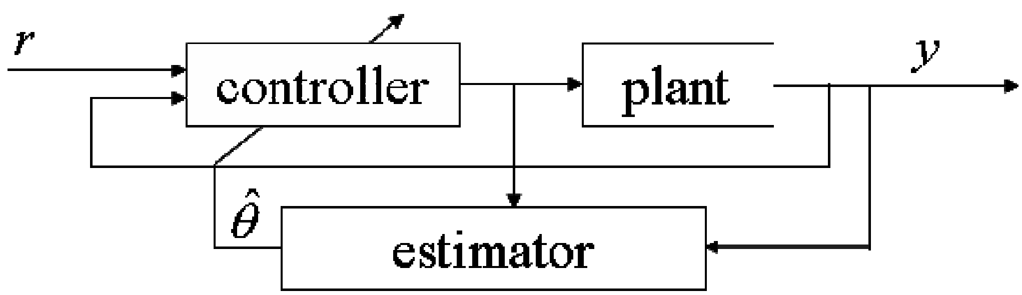

Figure 7.2: Self-Tuning Control System

The STC system combines a controller with an on-line (recursive) plant parameter estimator. A reference model can be added to the architecture. Basically, STC system performs simultaneous parameter identification and control. The controller parameters are computed from the estimates of the unknown plant parameters, as they were the true ones. This idea is often referred to as the *Certainty Equivalence Principle*. By coupling different control and estimation schemes, one can obtain a variety of self-tuning regulators.

When the true plane parameters are unknown, the controller parameters are either estimated directly (*direct* schemes) or computed by solving the same design equations using plant parameter estimates (*indirect* schemes). MRAC and STC systems can be designed using both direct and indirect approaches.

Our focus will be on the design, analysis and evaluation of the direct MRAC systems for continuous plants with uncertain dynamics.

Tracking Control Problem

In particular, we consider tracking problems for continuous plants operating in the presence of modeling uncertainties, environmental disturbances, and control failures. State feedback / feedforward tracking control will be designed for uncertain dynamical systems in the form

$$\begin{aligned} \dot{x} &= f(t, x, u, \Theta) \\ y &= h(x, \Theta) \end{aligned} \tag{7.1}$$

where x is the state, u is the control, Θ is a vector of unknown constant parameters, y is the controlled output. It is assumed that the system state vector x is available (measured on-line).

The tracking problem is to design the control input u so that the controlled output $y(t)$ tracks a given reference signal $r(t)$ in the presence of the system uncertainties, that is the *output tracking error*

$$e_y(t) = y(t) - r(t) \tag{7.2}$$

becomes sufficiently small, as $t \to \infty$. Moreover, it is required that during tracking, all the signals in the corresponding closed-loop system remain bounded.

If $e_y(t) \to 0$ then we say that an *asymptotic output tracking* is achieved. In general, it might not be feasible to achieve asymptotic tracking. In that case, the goal will be to achieve ultimate boundedness of the tracking error within a prescribed tolerance, that is

$$\left\| e_y(t) \right\| \le \varepsilon, \quad \forall\, t \ge T \tag{7.3}$$

where ε is the prescribed small positive number.

MRAC Design of 1st Order Systems

Suppose that a plant contains unknown constant parameters, without any information about their bounds. The plant dynamics is

$$\dot{x} = a\,x + b\big(u + f(x)\big) \tag{7.4}$$

where x is the state, u is the control input, a and b are <u>*unknown constants*</u>. It is assumed that the sign of b is known, while the unknown and possibly nonlinear function $f(x)$ is linearly parameterized in terms of N <u>*unknown constant parameters*</u> θ_i and <u>*known bounded basis functions*</u> φ_i.

$$f(x) = \sum_{i=1}^{N} \theta_i\,\varphi_i(x) = \theta^T\,\Phi(x) \tag{7.5}$$

In (7.5), $\Phi(x) = \big(\varphi_i(x) \;\; \ldots \;\; \varphi_i(x)\big)^T \in R^N$ denotes the known <u>*regressor vector*</u>. It is assumed that the regressor components $\varphi_i(x)$ are <u>*piece-wise continuous*</u> functions of the system state x.

A <u>*reference model*</u> is described by the 1st order differential equation

$$\dot{x}_m = a_m\,x_m + b_m\,r(t) \tag{7.6}$$

where $a_m < 0$ and b_m are the desired constants and $r(t)$ is the reference input.

<u>*The task*</u> is to design a control law $u(t)$ such that all the signals in the system remain bounded, while the <u>*tracking error*</u> $e(t) = x(t) - x_m(t)$ tends to zero asymptotically, as $t \to \infty$. Notice, that the tracking task must be accomplished in the presence of $(N+2)$ unknown constant parameters: $\{a,\, b,\, \theta_1,\, \ldots\,,\, \theta_N\}$.

First, we define an <u>*ideal control*</u> solution, as if the unknown parameters were known. The ideal control is formed using feedback / feedforward architecture

$$u_{ideal} = k_x\,x + k_r\,r - \theta^T\,\Phi(x) \tag{7.7}$$

Substituting (7.7) into (7.4), the closed-loop dynamics can be written.

$$\dot{x} = (a + b\,k_x)\,x + b\,k_r\,r(t) \tag{7.8}$$

Comparing (7.8) with the desired reference model dynamics (7.6), it immediately follows that ideal gains k_x and k_r must satisfy the following <u>*matching conditions*</u>

$$\begin{aligned} a + b\,k_x &= a_m \\ b\,k_r &= b_m \end{aligned} \tag{7.9}$$

Since in (7.9) there are 2 equations and two unknowns, it becomes clear that the ideal solution (which is not known !) always exists.

Based on (7.7), tracking control solution is formed.

$$u = \hat{k}_x\, x + \hat{k}_r\, r - \hat{\theta}^T\, \Phi(x) \tag{7.10}$$

where the feedback gain $\hat{k}_x$, the feedforward gain $\hat{k}_r$, and the estimated vector of parameters $\hat{\theta}$ will be found to achieve the desired tracking.

Towards this end, substitute (7.10) into the system dynamics (7.4). Then the closed-loop system becomes

$$\dot{x} = \left(a + b\,\hat{k}_x\right)x + b\left(\hat{k}_r\, r - \left(\hat{\theta} - \theta\right)^T \Phi(x)\right) \tag{7.11}$$

Using matching conditions (7.9) yields

$$\dot{x} = a_m\, x + \underbrace{b\,k_r}_{b_m} r + b\left(\hat{k}_x - k_x\right)x + b\left(\hat{k}_r - k_r\right)r - b\left(\hat{\theta} - \theta\right)^T \Phi(x) \tag{7.12}$$

Define the <u>*parameter estimation errors*</u> to be

$$\begin{aligned}\Delta k_x &= \hat{k}_x - k_x \\ \Delta k_r &= \hat{k}_r - k_r \\ \Delta\theta &= \hat{\theta} - \theta\end{aligned} \tag{7.13}$$

Then the <u>*closed-loop dynamics of the tracking error*</u> signal $e(t) = x(t) - x_m(t)$ can be obtained by subtracting (7.6) from (7.12).

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t) = a_m\, e + b\left(\Delta k_x\, x + \Delta k_r\, r - \Delta\theta^T\, \Phi(x)\right) \tag{7.14}$$

Consider the Lyapunov function (candidate).

$$V\left(e,\, \Delta k_x,\, \Delta k_r,\, \Delta\theta\right) = e^2 + |b|\left(\gamma_x^{-1}\, \Delta k_x^2 + \gamma_r^{-1}\, \Delta k_r^2 + \Delta\theta^T\, \Gamma_\theta^{-1}\, \Delta\theta\right) \tag{7.15}$$

where $\gamma_x > 0$, $\gamma_r > 0$, and $\Gamma_\theta = \Gamma_\theta^T > 0$ are the so-called <u>*rates of adaptation*</u>.
Taking the time derivative of V along the trajectories of (7.14), one gets

$$\dot{V}(e,\Delta k_x,\Delta k_r,\Delta\theta) = 2e\dot{e} + 2|b|\left(\gamma_x^{-1}\,\Delta k_x\,\dot{\hat{k}}_x + \gamma_r^{-1}\,\Delta k_r\,\dot{\hat{k}}_r + \Delta\theta^T\,\Gamma_\theta^{-1}\,\dot{\hat{\theta}}\right)$$
$$= 2e\left(a_m\,e + b\left(\Delta k_x\,x + \Delta k_r\,r - \Delta\theta^T\,\Phi(x)\right)\right) + 2|b|\left(\gamma_x^{-1}\,\Delta k_x\,\dot{\hat{k}}_x + \gamma_r^{-1}\,\Delta k_r\,\dot{\hat{k}}_r + \Delta\theta^T\,\Gamma_\theta^{-1}\,\dot{\hat{\theta}}\right)$$
$$= 2a_m\,e^2 + 2|b|\left(\Delta k_x\left(x\,e\,\mathrm{sgn}(b) + \gamma_x^{-1}\,\dot{\hat{k}}_x\right)\right) \tag{7.16}$$
$$+2|b|\left(\Delta k_r\left(r\,e\,\mathrm{sgn}(b) + \gamma_r^{-1}\,\dot{\hat{k}}_r\right)\right) + 2\,|b|\,\Delta\theta^T\left(-\Phi(x)\,e\,\mathrm{sgn}(b) + \Gamma_\theta^{-1}\,\dot{\hat{\theta}}\right)$$

Using (7.16), the *adaptive laws are chosen to enforce closed-loop stability*.

$$\begin{aligned}\dot{\hat{k}}_x &= -\gamma_x\,x\,e\,\mathrm{sgn}(b)\\ \dot{\hat{k}}_r &= -\gamma_r\,r\,e\,\mathrm{sgn}(b)\\ \dot{\hat{\theta}} &= \Gamma_\theta\,\Phi(x)\,e\,\mathrm{sgn}(b)\end{aligned} \tag{7.17}$$

In fact, due to (7.17) the time derivative becomes negative semi-definite, that is

$$\dot{V}(e,\,\Delta k_x,\,\Delta k_r,\,\Delta\theta) = 2\underbrace{a_m}_{<0}\,e(t)^2 \le 0 \tag{7.18}$$

which immediately implies that the signals $e,\,\Delta k_x,\,\Delta k_r,\,\Delta\theta$ are uniformly bounded. The latter coupled with the fact that $x_m(t),\ r(t)$ are bounded and θ is a constant vector, implies that the system state $x(t)$ and the estimated vector of parameters $\hat{\theta}(t)$ are uniformly bounded. It was assumed that the vector of the components $\varphi_i(x)$ of the regressor vector $\Phi(x)$ were piece-wise continuous functions of x. Therefore, they are uniformly bounded. Hence, the control $u(t)$ in (7.10) is uniformly bounded. Consequently, both $\dot{x}(t)$ and $\dot{x}_m(t)$ are uniformly bounded.

Differentiating (7.18) yields

$$\ddot{V}(e,\,\Delta k_x,\,\Delta k_r,\,\Delta\theta) = 4\,a_m\,e(t)\,\dot{e}(t) \tag{7.19}$$

Therefore $\ddot{V}$ is bounded and, consequently, $\dot{V}$ is uniformly continuous function of time. Since V is lower bounded, $\dot{V}$ is negative semi-definite and uniformly continuous, then all the three conditions of the Lyapunov-like lemma (Corollary 6.2) are satisfied, and therefore

$$\lim_{t\to\infty}\dot{V}(t) = 0 \tag{7.20}$$

Due to (7.18), we can finally conclude that the tracking error goes to zero asymptotically, as $t \to \infty$. Moreover, since the Lyapunov function is radially unbounded, the control solution is global, that is the closed-loop *tracking error dynamics is globally asymptotically stable*. The tracking problem is solved.

Theorem 7.1
For the uncertain dynamical system in (7.4) with the controller in (7.10) and the adaptive laws in (7.17), the closed-loop state $x(t)$ asymptotically tracks the state $x_m(t)$ of the reference model in (7.6), while all the signals in the closed-loop system remain bounded. Moreover, the closed-loop tracking error dynamics in (7.14) is globally asymptotically stable.

Lecture 5

8. Dynamic Inversion based MRAC Design for 1st Order Systems

Using similar design approach a dynamic inversion (DI) based adaptive control laws can be derived. Consider the uncertain dynamical system

$$\dot{x} = a\,x + b\,u + f(x) \tag{8.1}$$

Let the constants a and b be unknown. Assume that $b \ge b_0 > 0$, where b_0 is the known lower bound of b. Also assume that the unknown possible nonlinear function $f(x)$ is linearly parameterized in terms of the unknown constants θ_i and known *bounded* basis functions $\varphi_i(x)$, that is:

$$f(x) = \sum_{i=1}^{N} \theta_i\,\varphi_i(x) = \theta^T\,\Phi(x) \tag{8.2}$$

Let the reference model dynamics be specified as:

$$\begin{aligned} \dot{x}_m &= a_m\,x_m + b_m\,r(t) \\ a_m &< 0, \quad |r(t)| < \infty \end{aligned} \tag{8.3}$$

Rewrite the system dynamics in the form:

$$\dot{x} = \hat{a}\,x + \hat{b}\,u + \hat{f}(x) - \underbrace{(\hat{a} - a)}_{\Delta a}x - \underbrace{(\hat{b} - b)}_{\Delta b}u - \underbrace{\left(\hat{f}(x) - f(x)\right)}_{\Delta f(x)} \tag{8.4}$$

where $\hat{a}$, $\hat{b}$, and $\hat{f}(x) = \theta^T \Phi(x)$ represent the on-line estimated quantities, while Δa, Δb, and $\Delta f(x)$ are the corresponding *approximation errors*. Using

$$\hat{f}(x) = \sum_{i=1}^{N} \hat{\theta}_i \varphi_i(x) = \hat{\theta}^T \Phi(x) \tag{8.5}$$

the function approximation error can be written as:

$$\Delta\hat{f}(x) = \hat{f}(x) - f(x) = \sum_{i=1}^{N} \underbrace{\left(\hat{\theta}_i - \theta_i\right)}_{\Delta\theta_i} \varphi_i(x) = \Delta\theta^T \Phi(x) \tag{8.6}$$

Consider the following *dynamic inversion based adaptive controller*

$$u = \frac{1}{\hat{b}}\left(\left(a_m - \hat{a}\right)x + b_m r - \hat{\theta}^T \Phi(x)\right) \tag{8.7}$$

Substituting (8.7) into the 2nd term of (8.4), yields

$$\dot{x} = a_m x + b_m r - \Delta a\, x - \Delta b\, u - \Delta\theta^T \Phi(x) \tag{8.8}$$

Let $e = x - x_m$ be the tracking error signal. Its dynamics can be obtained by subtracting (8.3) from (8.8).

$$\dot{e} = a_m e - \Delta a\, x - \Delta b\, u - \Delta\theta^T \Phi(x) \tag{8.9}$$

Consider the following Lyapunov function candidate:

$$V(e,\, \Delta a,\, \Delta b,\, \Delta\theta) = e^2 + \gamma_a^{-1} \Delta a^2 + \gamma_b^{-1} \Delta b^2 + \Delta\theta^T \Gamma_\theta^{-1} \Delta\theta \tag{8.10}$$

where $\gamma_a > 0$, $\gamma_b > 0$, $\Gamma_\theta = \Gamma_\theta^T > 0$ will eventually become the adaptation rates. The timed derivative of V along the trajectories of the error dynamics (8.9) can be computed:

$$\begin{aligned}
\dot{V}(e,\, \Delta a,\, \Delta b,\, \Delta\theta) &= 2e\dot{e} + 2\left(\gamma_a^{-1} \Delta a\, \dot{\hat{a}} + \gamma_b^{-1} \Delta b\, \dot{\hat{b}} + \Delta\theta^T \Gamma_\theta^{-1} \dot{\hat{\theta}}\right) \\
&= 2e\left(a_m e - \Delta a\, x - \Delta b\, u - \Delta\theta^T \Phi(x)\right) + 2\left(\gamma_a^{-1} \Delta a\, \dot{\hat{a}} + \gamma_b^{-1} \Delta b\, \dot{\hat{b}} + \Delta\theta^T \Gamma_\theta^{-1} \dot{\hat{\theta}}\right) \\
&= 2a_m e^2 + \Delta a\left(\gamma_a^{-1} \dot{\hat{a}} - x e\right) + \Delta b\left(\gamma_b^{-1} \dot{\hat{b}} - u e\right) + \Delta\theta^T \left(\Gamma_\theta^{-1} \dot{\hat{\theta}} - \Phi(x) e\right)
\end{aligned} \tag{8.11}$$

Based on (8.11) and in order to make $\dot{V} \le 0$, the adaptive laws are chosen as:

$$\dot{\hat{a}} = \gamma_a \, x \, e$$
$$\dot{\hat{b}} = \gamma_b \, u \, e \tag{8.12}$$
$$\dot{\hat{\theta}} = \Gamma_\theta \, \Phi(x) e$$

In fact, this leads to

$$\dot{V}(e,\, \Delta a,\, \Delta b,\, \Delta\theta) = 2\, a_m \, e^2 \le 0 \tag{8.13}$$

Therefore, the signals $e,\, \Delta a,\, \Delta b,\, \Delta\theta$ are bounded. Since $r(t)$ is bounded, then the reference model state x_m is bounded. Hence, $x,\, \hat{a},\, \hat{b},\, \hat{\theta}$ are bounded.

Due to the division by $\hat{b}$ in (8.7) and in order to keep the control signal u bounded, we need to modify adaptive laws (8.12). Consider the following modification of the 2nd equation in (8.12):

$$\dot{\hat{b}} = \begin{cases} \gamma_b \, u \, e, & \text{if } \hat{b} \ge b_0 \vee \left[\hat{b} = b_0 \wedge (u\, e) > 0 \right] \\ 0, & \text{if } \hat{b} = b_0 \wedge (u\, e) < 0 \end{cases} \tag{8.14}$$

Basically, the intent is to stop the adaptation if the $\hat{b}$ reaches its lower limit b_0 and its time derivative is negative. One needs to verify that this modification does not adversely effects the closed-loop stability. Formally, we need to show that

$$\Delta b \left(\gamma_b^{-1} \dot{\hat{b}} - u\, e \right) \le 0 \tag{8.15}$$

When $\hat{b} \ge b_0$, the adaptive law (8.14) is the same as the corresponding law in (8.12) and, therefore $\dot{V} \le 2\, a_m \, e^2 \le 0$. Suppose that there exists T such that $\hat{b}(T) = b_0$. Since $b \ge b_0$ then $\Delta b(T) = \hat{b}(T) - b = b_0 - b \le 0$. If $u(T) e(T) \ge 0$ then again $\dot{V} = 2\, a_m \, e^2 \le 0$, while $\dot{\hat{b}}(T) = \gamma_b \, u \, e \ge 0$ implying that $\hat{b}(t)$ increases locally for $t \ge T$. On other hand, if $u(T) e(T) < 0$ then according to (8.14), $\Delta b \left(\gamma_b^{-1} \dot{\hat{b}} - u\, e \right) = -\underbrace{\Delta b}_{\le 0} \underbrace{u\, e}_{\le 0} \le 0$. As a result, $\dot{V} \le 2\, a_m \, e^2 \le 0$. Thus, modification (8.14) always contributes to making the time derivative of V to be negative-semidefinite.

The adaptive laws can now be written explicitly.

$$\begin{cases} \dot{\hat{a}} = \gamma_a \, x\left(x - x_m\right) \\ \dot{\hat{b}} = \begin{cases} \gamma_b \, u\left(x - x_m\right), & \text{if } \hat{b} \geq b_0 \vee \left[\hat{b} = b_0 \wedge \left(u\left(x - x_m\right)\right) > 0\right] \\ 0, & \text{if } \hat{b} = b_0 \wedge \left(u\, e\right) < 0 \end{cases} \\ \dot{\hat{\theta}} = \Gamma_\theta \, \Phi\left(x\right)\left(x - x_m\right) \end{cases} \tag{8.16}$$

Next, a formal proof is given to show that the DI based adaptive control in (8.7) provides asymptotic tracking of the reference model state.

Since $\dot{V} \leq 0$ then $e,\ \Delta a,\ \Delta b,\ \Delta\theta$ are bounded. The latter implies that $x,\ \hat{a},\ \hat{b},\ \hat{\theta}$ are bounded. Due to modification (8.14), $\hat{b} \geq b_0$ and consequently u is bounded. This in turn implies that $\dot{x}$ is bounded. Moreover, since r is bounded, then $\dot{x}_m$ is bounded and therefore $\dot{e}$ is bounded. Because of (8.14)

$$\dot{V}\left(e,\ \Delta a,\ \Delta b,\ \Delta\theta\right) \leq -2\left|a_m\right| e^2 \leq 0 \tag{8.17}$$

for all $t \geq 0$. Since V is bounded from below by zero and its derivative is semi-negative, V converges to a limit, as a function of time. Integrating both sides of (8.17) yields:

$$V\left(t\right) - V\left(0\right) \leq -2\left|a_m\right| \int_0^t e^2\left(\tau\right) d\tau \leq 0 \tag{8.18}$$

or, equivalently:

$$\int_0^t e^2\left(\tau\right) d\tau \leq \frac{1}{2\left|a_m\right|}\left(V\left(0\right) - V\left(t\right)\right) < \infty \tag{8.19}$$

Let $W\left(t\right) = \int_0^t e^2\left(\tau\right) d\tau$. From (8.19) it follows that $W\left(t\right)$ tends to a finite limit, as $t \to \infty$. At the same time its time derivative is $\dot{W}\left(t\right) = e^2\left(t\right)$. Since $\ddot{W}\left(t\right) = 2\, e\, \dot{e}\left(t\right) < \infty$ then $\dot{W}\left(t\right)$ is uniformly continuous. Using Barbalat's Lemma, implies that $\lim_{t\to\infty} \dot{W}\left(t\right) = 0$. Thus, $\lim_{t\to\infty} e^2\left(t\right) = 0$ and the tracking problem is solved.

Remark 8.1

Modification (8.14) is a special case of the well-known Projection Operator. Since the right hand side of (8.14) is not Lipschitz the closed-loop system does not satisfy the sufficient conditions to have a unique trajectory, given an initial state. Corresponding solutions can be defined similar to the case of variable structure systems such as systems with sliding modes. Nevertheless, a continuous version of the Projection Operator exists and will be covered later in the course.

Lecture 6

9. MRAC Design for Affine-in-Control MIMO Systems

Reading material:
[1]: Chapter 8, Section 8.3
[1]: Chapter 8, Section 8.5

In this section, we consider MRAC design for a class of multi-input-multi-output (MIMO) nonlinear systems whose plant dynamics is linearly parameterized, the uncertainties satisfy the so-called matching conditions, and if the full state is measurable, (i.e., available on-line as the system output). More specifically, consider the n^{th} order MIMO system in the form:

$$\dot{x} = A\,x + B\,\Lambda\left(u + f\left(x\right)\right) \tag{9.1}$$

where $x \in R^n$ is the system state, $u \in R^m$ is the control input, $B \in R^{n\times m}$ is known matrix, $A \in R^{n\times n}$ and $\Lambda \in R^{m\times m}$ are unknown matrices. In addition, it is assumed that Λ is diagonal, its elements λ_i are non-negative, and the pair $\left(A,\ B\,\Lambda\right)$ is controllable. The uncertainty in Λ is introduced to model a control failure phenomenon.

Moreover, the unknown possibly nonlinear function $f\left(x\right): R^n \to R^m$ represents the so-called *system matched uncertainty*. It is assumed that the function can be written as a linear combination of N known *bounded* basis functions with unknown constant coefficients.

$$f\left(x\right) = \Theta^T\,\Phi\left(x\right) \tag{9.2}$$

In (9.2), $\Theta \in R^{N\times m}$ is the unknown constant matrix, while $\Phi\left(x\right) \in R^N$ represents the known regressor vector.

The *control objective* of the *MIMO tracking problem* is to choose the input vector u such that all signals in the closed-loop system are bounded and the state x follows the state $x_{ref} \in R^n$ of a reference model specified by the LTI system

$$\dot{x}_{ref} = A_{ref}\, x_{ref} + B_{ref}\, r(t) \tag{9.3}$$

where $A_{ref} \in R^{n \times n}$ is Hurwitz, $B_{ref} \in R^{n \times m}$, and $r(t) \in R^{m}$ is a bounded reference input vector. Note that the reference model and its external input $r(t)$ must be chosen so that $x_{ref}(t)$ represents a ***desired*** trajectory that $x(t)$ has to follow. In other words, the control input u needs to be chosen such that the tracking error vector asymptotically tends to zero.

$$\lim_{t \to \infty} \left\| x(t) - x_m(t) \right\| = 0 \tag{9.4}$$

If the matrices A and Λ were known, one could apply the control law

$$u = K_x^T\, x + K_r^T\, r - \Theta^T\, \Phi(x) \tag{9.5}$$

and obtain the closed-loop system

$$\dot{x} = \left(A + B\,\Lambda\, K_x^T\right) x + B\,\Lambda\, K_r^T\, r \tag{9.6}$$

Comparing (9.6) with the desired dynamics in (9.3), it follows that the ideal (unknown) matrix gains must be chosen to satisfy the so-called ***matching conditions***:

$$\begin{cases} A + B\,\Lambda\, K_x^T = A_{ref} \\ B\,\Lambda\, K_r^T = B_{ref} \end{cases} \tag{9.7}$$

Assuming that the matching conditions take place, it is easy to see that the closed-loop system is the same as that of the reference model, and consequently, asymptotic (exponential) tracking is achieved for any bounded reference input signal $r(t)$.

Remark 9.1
Given the matrices $A, B, \Lambda, A_{ref}, B_{ref}$, no K_x, K_r may exist to satisfy the matching conditions (9.7) indicating that the control law (9.5) may not have enough structural flexibility to meet the control objective. Often in practice, the structure of A is known, and the reference model matrices A_{ref}, B_{ref} are chosen so that (9.7) has a solution for K_x, K_r.

Assuming that K_x, K_r in (9.7) exist, consider the following control law:

$$u = \hat{K}_x^T\, x + \hat{K}_r^T\, r - \hat{\Theta}^T\, \Phi(x) \tag{9.8}$$

where $\hat{K}_x \in R^{n\times m}$, $\hat{K}_r \in R^{m\times m}$, $\hat{\Theta} \in R^{N\times n}$ are the estimates of the ideal unknown matrices K_x, K_r, Θ, respectively. The estimated matrices will be generated on-line and by an appropriate adaptive law.

Substituting (9.8) into (9.1), the closed-loop system dynamics can be written.

$$\dot{x} = \left(A + B\Lambda\hat{K}_x^T\right)x + B\Lambda\left(\hat{K}_r^T r - \left(\hat{\Theta}-\Theta\right)^T \Phi(x)\right) \tag{9.9}$$

Subtracting (9.3) from (9.9), closed-loop dynamics of the n – dimensional tracking error vector $e(t) = x(t) - x_{ref}(t)$ can be obtained.

$$\dot{e} = \left(A + B\Lambda\hat{K}_x^T\right)x + B\Lambda\left(\hat{K}_r^T r - \left(\hat{\Theta}-\Theta\right)^T \Phi(x)\right) - A_{ref}\, x_{ref} - B_{ref}\, r \tag{9.10}$$

Using matching conditions (9.7) further yields:

$$\begin{aligned}\dot{e} &= \left(A_{ref} + B\Lambda\left(\hat{K}_x - K_x\right)\right)x - A_{ref}\, x_{ref} + B\Lambda\left(\hat{K}_r - K_r\right)r - B\Lambda\left(\hat{\Theta}-\Theta\right)^T \Phi(x) \\ &= A_{ref}\, e + B\Lambda\left[\left(\hat{K}_x - K_x\right)^T x + \left(\hat{K}_r - K_r\right)^T r - \left(\hat{\Theta}-\Theta\right)^T \Phi(x)\right]\end{aligned} \tag{9.11}$$

Let $\Delta K_x = \hat{K}_x - K_x$, $\Delta K_r = \hat{K}_r - K_r$, and $\Delta\Theta = \hat{\Theta} - \Theta$ represent the ***parameter estimation errors***. In terms of the latter, the tracking error dynamics becomes:

$$\dot{e} = A_{ref}\, e + B\Lambda\left[\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right] \tag{9.12}$$

Vector and matrix norms

Before proceeding any further, recall that given a matrix $A = \left[a_{ij}\right] \in R^{n\times m}$, the ***Frobenius norm*** is defined by

$$\|A\|_F = \mathrm{tr}\left(A^T A\right) = \sum_{i,j} a_{ij}^2 \tag{9.13}$$

with $\mathrm{tr}(\;)$ the trace operator. On the other hand, given any vector *p*-norm, the ***induced matrix norm*** is defined by

$$\|A\|_p = \sup_{x\neq 0} \frac{\|A x\|_p}{\|x\|_p} \tag{9.14}$$

Collection of Facts about vector and matrix norms, (prove it).

- For vector 1-norm $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, the induced matrix norm is equal to the maximum absolute column sum, that is: $\|A\|_1 = \max_{1 \le j \le m} \left| \sum_{i=1}^{n} a_{ij} \right|$.
- For vector 2-norm $\|x\|_2 = \sum_{i=1}^{n} x_i^2$, the induced matrix norm is equal to the maximum singular value of A, that is: $\|A\|_2 = \sigma_{\max}(A)$.
- For vector ∞-norm $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$, the induced matrix norm is equal to the maximum absolute row sum, that is: $\|A\|_\infty = \max_{1 \le i \le n} \left| \sum_{j=1}^{m} a_{ij} \right|$.
- The induced matrix norm satisfies: $\|A x\|_p \le \|A\|_p \|x\|_p$, and for any two compatibly dimensioned matrices, A and B, one also has: $\|A B\|_p \le \|A\|_p \|B\|_p$.
- The Frobenius norm *is not* an induced norm of any vector norm, but it is compatible with the 2-norm in the sense that: $\|A x\|_2 \le \|A\|_F \|x\|_2$.
- For any two compatibly dimensioned matrices A and B, the *Frobenius inner product* is defined as: $\langle A, B \rangle_F = A^T B$.
- According to the Schwartz inequality one has: $\left\| \langle A, B \rangle_F \right\|_F = \left\| A^T B \right\|_F \le \|A\|_F \|B\|_F$.
- For any two co–dimensional vectors a and b, the *trace identity* takes place: $a^T b = \operatorname{tr}\left(b a^T\right)$.

Let $\Gamma_x = \Gamma_x^T > 0, \quad \Gamma_r = \Gamma_r^T > 0, \quad \Gamma_\Theta = \Gamma_\Theta^T > 0$. Going back to analyzing the tracking error dynamics in (9.12), consider the Lyapunov function candidate:

$$V(e, \Delta K_x, \Delta K_r, \Delta\Theta) = e^T P e + \operatorname{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \Delta K_x + \Delta K_r^T \Gamma_r^{-1} \Delta K_r + \Delta\Theta^T \Gamma_\Theta^{-1} \Delta\Theta\right] \Lambda\right) \quad (9.15)$$

where $P = P^T > 0$ satisfies the *algebraic Lyapunov equation*

$$P A_{ref} + A_{ref}^T P = -Q \quad (9.16)$$

for some $Q = Q^T > 0$. Then the time derivative of V, evaluated along the trajectories of (9.12), can be calculated.

$$
\begin{aligned}
\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}}\right]\Lambda\right) \\
&= \left(A_{ref}\, e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right)\right)^T P e \\
&+ e^T P\left(A_{ref}\, e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right)\right) \\
&+ 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}}\right]\Lambda\right) \\
&= e^T \left(A_{ref}\, P + P\, A_{ref}\right) e + 2 e^T P B \Lambda\left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right) \\
&+ 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}}\right]\Lambda\right)
\end{aligned}
\tag{9.17}
$$

Using (9.16), yields:

$$
\begin{aligned}
\dot{V} &= -e^T Q e + \left[2 e^T P B \Lambda \Delta K_x^T x + 2\,\mathrm{tr}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x \Lambda\right)\right] \\
&+ \left[2 e^T P B \Lambda \Delta K_r^T r + 2\,\mathrm{tr}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r \Lambda\right)\right] \\
&+ \left[-2 e^T P B \Lambda \Delta\Theta^T \Phi(x) + 2\,\mathrm{tr}\left(\Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} \Lambda\right)\right]
\end{aligned}
\tag{9.18}
$$

Using the trace identity, one gets

$$
\left[
\begin{aligned}
&\underbrace{e^T P B \Lambda}_{a^T} \underbrace{\Delta K_x^T x}_{b} = \mathrm{tr}\left(\underbrace{\Delta K_x^T x}_{b} \underbrace{e^T P B \Lambda}_{a^T}\right) \\
&\underbrace{e^T P B \Lambda}_{a^T} \underbrace{\Delta K_r^T r}_{b} = \mathrm{tr}\left(\underbrace{\Delta K_r^T r}_{b} \underbrace{e^T P B \Lambda}_{a^T}\right) \\
&\underbrace{e^T P B \Lambda}_{a^T} \underbrace{\Delta\Theta^T \Phi(x)}_{b} = \mathrm{tr}\left(\underbrace{\Delta\Theta^T \Phi(x)}_{b} \underbrace{e^T P B \Lambda}_{a^T}\right)
\end{aligned}
\right.
\tag{9.19}
$$

Substituting (9.19) into (9.18), further yields

$$
\begin{aligned}
\dot{V} &= -e^T Q e + 2\,\mathrm{tr}\left(\Delta K_x^T \left[\Gamma_x^{-1} \dot{\hat{K}}_x + x e^T P B\right]\Lambda\right) \\
&+ 2\,\mathrm{tr}\left(\Delta K_r^T \left[\Gamma_r^{-1} \dot{\hat{K}}_r + r e^T P B\right]\Lambda\right) + 2\,\mathrm{tr}\left(\Delta\Theta^T \left[\Gamma_\Theta^{-1} \dot{\hat{\Theta}} - \Phi(x) e^T P B\right]\Lambda\right)
\end{aligned}
\tag{9.20}
$$

Adaptive laws are chosen as follows:

$$\begin{cases} \dot{\hat{K}}_x = -\Gamma_x \, x e^T P B \\ \dot{\hat{K}}_r = -\Gamma_r \, r(t) e^T P B \\ \dot{\hat{\Theta}} = \Gamma_\Theta \, \Phi(x) e^T P B \end{cases} \tag{9.21}$$

Then the time-derivative of V becomes negative semi-definite.

$$\dot{V} = -e^T Q e \le 0 \tag{9.22}$$

Therefore the closed-loop error dynamics is stable, that is the tracking error $e(t)$ and the parameter estimation errors $\Delta K_x(t)$, $\Delta K_r(t)$, $\Delta\Theta(t)$ are bounded signals in time. Therefore, the parameter estimates $\hat{K}_x(t)$, $\hat{K}_r(t)$, $\hat{\Theta}(t)$ are also bounded. Since $r(t)$ is bounded then $x_{ref}(t)$ and $\dot{x}_{ref}(t)$ are bounded. Hence, the system state $x(t)$ is bounded and, consequently the control input $u(t)$ in (9.8) is bounded. The latter implies that $\dot{x}(t)$ is bounded and, hence $\dot{e}(t)$ is bounded. Furthermore, the 2$^{\text{nd}}$ time derivative of $V(t)$

$$\ddot{V} = -e^T Q e = -2 e^T Q \dot{e} \tag{9.23}$$

is bounded and thus $\dot{V}(t)$ is a uniformly continuous function of time. The latter coupled with the facts that $V(t)$ is lower bounded and $\dot{V}(t) \le 0$ implies (Barbalat's Lemma) that $\lim_{t\to\infty} \dot{V}(t) = 0$. Thus $\lim_{t\to\infty} \|e(t)\| = 0$ and the MIMO tracking problem is solved.

Remark 9.2 (prove it)
If some of the diagonal elements λ_i of the unknown diagonal matrix Λ are negative and the signs of all of them are known, then the adaptive laws

$$\begin{cases} \dot{\hat{K}}_x = -\Gamma_x \, x e^T P B \operatorname{sgn}\Lambda \\ \dot{\hat{K}}_r = -\Gamma_r \, r(t) e^T P B \operatorname{sgn}\Lambda \\ \dot{\hat{\Theta}} = \Gamma_\Theta \, \Phi(x) e^T P B \operatorname{sgn}\Lambda \end{cases} \tag{9.24}$$

solve the MIMO tracking problem, where $\operatorname{sgn}\Lambda = \operatorname{diag}\left[\operatorname{sgn}\lambda_1, \ldots, \operatorname{sgn}\lambda_m\right]$.

References

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