Adaptive Control: CDS 270

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Lecture 1

1. Introduction

Reading material:
[1]: Chapter 1, Sections 1.1, 1.2
[1]: Chapter 3, Section 3.1
[2]: Chapter 1, Sections 1.1, 1.2.1

We consider dynamical systems that are modeled by a *finite* number of coupled 1st order ordinary differential equations (ODE-s):

$$\dot{x} = f(t, x, u) \tag{1.1}$$

In (1.1), t denotes time and f is a vector field. We call (1.1) the *state equation*, refer to $x \in R^n$ as the *system state*, and $u \in R^m$ as the *control input*, (external signal). The number of the state vector components n is called the *order* of the system. Sometimes, another equation

$$y = h(t, x, u) \tag{1.2}$$

is also given, where $y \in R^p$ denotes the *system output*. Equations (1.1) and (1.2) together form the *system state space model*.

A solution $x(t)$ of (1.1) (if one exists) corresponds to a curve in state space, as t varies from and initial time to infinity. This curve is often referred to as a *state trajectory* or a *system trajectory*.

A special case of (1.1) – (1.2) is linear (*affine in the control input*) system

$$\begin{aligned} \dot{x} &= f(t, x) + g(t, x)u \\ y &= h(t, x) \end{aligned} \tag{1.3}$$

Letting $x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T$, a special class of nonlinear continuous-time dynamics is given by systems in *Brunovsky canonical form*.

$$
\begin{aligned}
&\dot{x}_1 = x_2 \\
&\dot{x}_2 = x_3 \\
&\dots\dots\dots \\
&\dot{x}_n = f(x) + g(x)u \\
&y = h(x)
\end{aligned}
\tag{1.4}
$$

For *linear time-variant (LTV) systems* the state space model (1.1) – (1.2) is:

$$
\begin{aligned}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x + D(t)u
\end{aligned}
\tag{1.5}
$$

Finally, the class of *linear time-invariant (LTI) systems* is written in the familiar form:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
\tag{1.6}
$$

If (1.1) does not contain an input signal u

$$\dot{x} = f(t, x) \tag{1.7}$$

then the resulting dynamics is called *unforced*. If in addition the function f does not depend explicitly on t, that is

$$\dot{x} = f(x) \tag{1.8}$$

then the system dynamics is called *autonomous* or time-invariant. Systems that do depend on time (explicitly) are called *non-autonomous* or time-variant.

A point $x = x^*$ in the state space is an *equilibrium* point of (1.8) if

$$f(x^*) = 0 \tag{1.9}$$

In other words, whenever the state of the system starts at x^*, it will remain at x^* for all future times.

The linear system $\dot{x} = Ax$ has an ***isolated equilibrium*** point at $x = 0$ if $\det A \neq 0$, that is if A has no zero eigenvalues. Otherwise, the system has a continuum of equilibrium points. These are the only possible equilibrium patterns that a linear system may have. On the other hand, a nonlinear system (1.8) can have ***multiple*** isolated equilibrium points.

Lemma 1.1 (prove it!)
Trajectories of a 1st order ***autonomous*** ODE (assuming that they exist) are monotonic functions of time.

2. Existence and Uniqueness

Reading material:
[1]: Chapter 1, Section 4.10
[2]: Chapter 3, Section 3.1
[2]: Appendix A
[2]: Appendix C1

For the unforced system (1.7) to be a useful mathematical model of a physical system, it must be able to predict future states of the system given its current state x_0 at t_0. In other words, the ***Initial Value Problem*** (IVP)

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{2.1}$$

must have a ***unique*** solution.

Example 2.1
The IVP

$$\dot{x} = \begin{cases} -1 - x^2, & x \geq 0 \\ 1 + x^2, & x < 0 \end{cases}$$
$$x(0) = 0$$

has no solutions at all (show!) on the interval $0 \leq t \leq 1$.

Example 2.2
The IVP

$$\dot{x} = x^{\frac{2}{3}}$$
$$x(0) = 0$$

has *infinitely many* solutions (show!), each of which is defined on R:

$$x(t)=\begin{cases}\frac{1}{27}(t-a)^3\ , & t<a\\ 0\ , & a\le t\le b\\ \frac{1}{27}(t-b)^3\ , & t>b\end{cases}$$

where $a<0$ and $b>0$ are arbitrary constants. If $a=b=0$ then there are 2 solutions: $x(t)=\frac{t^3}{27}$ and $x(t)\equiv 0$.

The *existence* and *uniqueness* of IVP can be ensured by imposing appropriate constraints on the right hand side function $f(t,x)$ in (2.1).

We start by stating a *sufficient condition* for the IVP problem to admit a solution which may not be necessarily unique.

Theorem 2.1 (Cauchy / Peano Existence Theorem)
If $f(t,x)$ is continuous in a closed region

$$B=\{(t,x): |t-t_0|\le T,\ \|x-x_0\|\le R\}\subseteq R\times R^n \tag{2.2}$$

where T and R are strictly positive constants, then there exists $t_0<t_1\le T$ such that the IVP has at least one continuous in time solution $x(t)$.

In other words, continuity of $f(t,x)$ in its arguments ensures that there is at least one solution of the IVP in (2.1).

The above theorem does not guarantee the uniqueness of the solution. The key constraint that yields uniqueness is the *Lipschitz condition*, whereby $f(t,x)$ satisfies the inequality

$$\|f(t,x)-f(t,y)\|\le L\|x-y\| \tag{2.3}$$

for all (t,x) and (t,y) in some neighborhood of (t_0,x_0). Note that in (2.3), $\|\bullet\|$ denotes any *p*-norm.

The next theorem gives a *sufficient condition* for the *unique existence* of a solution.

Theorem 2.2 (Local Existence and Uniqueness)

Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition (2.3)

$$\forall\, x, y \in B = \left\{x \in R^n : \|x - x_0\| \le r\right\},\ \forall\, t \in [t_0, t_1] \tag{2.4}$$

Then, there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

The key assumption in the above theorem is the Lipschitz condition (2.3) which is assumed to be valid *locally*, that is in a neighborhood of (t_0, x_0) on the compact domain B in (2.4).

One may try to extend the interval of existence and uniqueness over a given time interval $[t_0, t_0 + \delta]$ by taking $t_0 \triangleq t_0 + \delta$ as a new initial time and $x_0 \triangleq x(t_0 + \delta)$ as a new initial state. If the conditions of the theorem are satisfied at $(t_0 + \delta,\ x(t_0 + \delta))$ then there exist $\delta_2 > 0$ such that the IVP has a unique solution over $[t_0 + \delta, t_0 + \delta + \delta_2]$ that passes through the point $(t_0 + \delta,\ x(t_0 + \delta))$. We piece together the solutions to establish the existence of a unique solution over the interval $[t_0, t_0 + \delta + \delta_2]$. This idea can be repeated to keep extending the solution. However, in general the solution cannot be extended indefinitely. In that case, there will be a maximum interval $[t_0, T)$, where the unique solution exists.

Example 2.3
The IVP

$$\dot{x} = x^2$$
$$x(0) = 1$$

has a solution

$$x(t) = -\frac{1}{t-1}$$

which is defined only for $t < 1$ and can not be extended to R. Note that the function $f(x) = x^2$ is locally Lipschitz for all $x \in R$, and as $t \to 1$ the solution has a finite escape time, that is it leaves any compact set within a finite time. The phrase “finite escape time” is used to describe the phenomenon that a trajectory escapes to infinity at a finite time.

Assuming that $f(x)$ is ***globally Lipschitz***, the next theorem establishes the existence of a unique solution over any arbitrarily large interval.

Theorem 2.3 (Global existence and Uniqueness)
Suppose that f is piecewise continuous in t and *globally* Lipschitz in x, that is the function satisfies Lipschitz condition (2.3)

$$\forall x, y \in R^n, \ \forall t \in [t_0, t_1] \tag{2.5}$$

Then the IVP (2.1) has a unique solution over $[t_0, t_1]$, where the final time t_1 may be arbitrarily large.

Sufficient conditions in the above theorem are overly conservative.

Example 2.4
The IVP

$$\begin{aligned} \dot{x} &= -x^3 \\ x(0) &= x_0 \end{aligned}$$

has a unique solution

$$x(t) = \frac{x_0}{\sqrt{2\, x_0^2\, t + 1}}$$

for any initial condition x_0 and for all $t \geq 0$.

Basically, if it is known that IVP has a solution that evolves on a compact domain then the solution can be extended indefinitely.

Theorem 2.4 (Global existence and Uniqueness on a compact domain)
Let $f(t, x)$ be piecewise continuous in t, locally Lipschitz in x for all $t \geq 0$ and all x in a domain $D \subset R^n$. Let $W \subset D$ be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of the corresponding IVP lies entirely in W. Then there is a unique solution that is defined for all $t \geq 0$.

Remark: There are extensions that deal with existence and uniqueness of IVP-s whose system dynamics is discontinuous in x, (i.e., not Lipschitz).

Example 2.5 (sliding mode, prove and simulate)
The IVP

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\operatorname{sgn}(x_1 + x_2) \end{cases}$$
$$x(0) = x_0$$

has a unique solution for any initial condition vector x_0 and for all $t \geq 0$. The solution reaches manifold $x_1 + x_2 = 0$ in finite time and "slides" down the manifold towards the origin.

3. Lyapunov Stability

Reading material:
[1]: Chapter 3, Sections 3.1-3.4
[1]: Chapter 3, Section 4.5
[2]: Chapter 4

Stability of equilibrium points is usually characterized in the sense of Lyapunov:

Alexander Michailovich Lyapunov, 1857-1918

- Russian mathematician and engineer who laid out the foundation of the Stability Theory
- Results published in 1892, Russia
- Translated into French, 1907
- Reprinted by Princeton University, 1947
- American Control Engineering Community Interest, 1960's

Lyapunov stability theorems give *sufficient conditions* for stability, asymptotic stability, and so on. Statements that establish necessity of these conditions are called the converse theorems.

For example, it is known that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization of the system about that point has an exponentially stable equilibrium at the origin.

We will be mostly concern with the 2nd theorem of Lyapunov. We will use it to: a) derive stable adaptive laws for uncertain system, and b) show boundedness of the system closed-loop solutions even when the system has no equilibrium points.

Without a loss of generality, we'll study *stability of the origin* for the autonomous system:

$$\dot{x} = f(x) \tag{3.1}$$

where $f(x)$ is locally Lipschitz in x and $f(0)=0$.

Definition 3.1 (local stability)
The equilibrium point $x=0$ of (3.1) is

- *stable* if
$$\forall R>0,\quad \exists\, r(R)>0,\quad \left\{\|x(0)\|<r\right\} \Rightarrow \left\{\forall\, t\geq 0, \|x(t)\|<R\right\} \tag{3.2}$$
- *unstable* if it is not stable (write formal definition similar to (3.2))
- *asymptotically stable* if it is stable and $r=r(R)$ can be chosen such that
$$\|x(0)\|<r \Rightarrow \lim_{t\to\infty}\|x(t)\|=0 \tag{3.3}$$
- *marginally stable* if it is stable but not asymptotically stable, (write formal definition)
- *exponentially stable* if it is stable and
$$\exists\, r,\alpha,\lambda>0,\quad \forall\left\{\|x(0)\|<r \wedge t>0\right\}:\quad \|x(t)\|\leq\alpha\|x(0)\|e^{-\lambda t}, \tag{3.4}$$

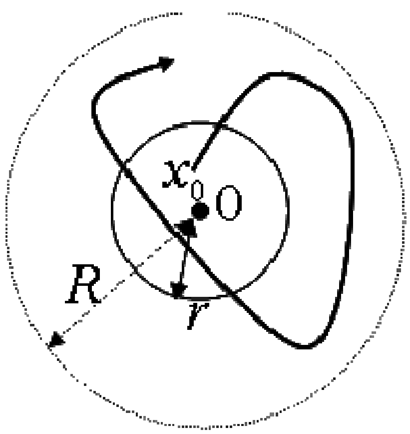

Figure 3.1: Stable System

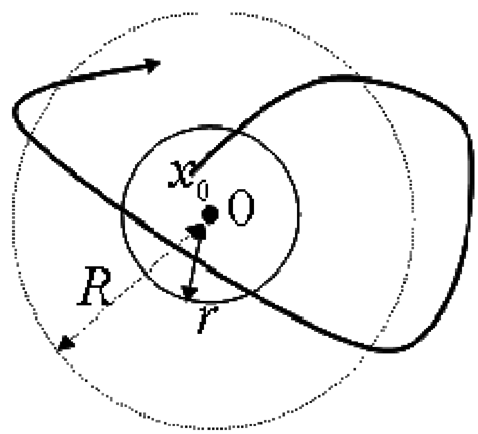

Figure 3.2: Unstable System

Basically, an equilibrium point is *stable* if all solutions starting at nearby points stay nearby; otherwise it is unstable. It is *asymptotically stable* if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Remark: Stabilizable systems are not necessarily stable.

Note that by definition, exponential stability implies asymptotic stability, which in turn implies stability.

By definition, stability in the sense of Lyapunov defines *local* behavior of the system trajectories near the equilibrium. In order to analyze how the system behaves some distance away from the equilibrium, global concepts of stability are required.

Definition 3.2 (global stability)

If asymptotic (exponential) stability holds for any initial states, the equilibrium point is said to be *globally* asymptotically (exponentially) stable.

Next, two main theorems of Lyapunov are presented.

Theorem 3.1 (Lyapunov indirect method)
Let $x = 0$ be an equilibrium point for the nonlinear system (3.1), where $f : D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0} \tag{3.5}$$

Then:

- the origin is asymptotically stable if $\operatorname{Re}\lambda_i < 0$ for all eigenvalues of A
- the origin is unstable if $\operatorname{Re}\lambda_i > 0$ for at least one of the eigenvalues of A
- if at least one of the eigenvalues is on the $j\omega$ axis, (i.e., the linearized system is marginally stable), then nothing can be said about the original nonlinear system behavior

Before stating the 2nd theorem of Lyapunov we need to introduce the concept of positive definite functions.

Definition 3.3
Let $D \subset R^n$ be a neighborhood of the origin. A function $V(x) : D \to R$ is said to be (positive semidefinite) if

- *locally positive definite*, if: $V(0) = 0$ and $V(x) > 0, \quad \forall\, x \in D - \{0\}$
- *locally positive semidefinite*, if: $V(0) = 0$ and $V(x) \geq 0, \quad \forall\, x \in D - \{0\}$
- *locally negative definite* (*semidefinite*), if it is not locally positive definite (semidefinite)

If in the above definition $D = R^n$ then the function is *globally* positive (negative) definite (semidefinite).

Theorem 3.2 (Lyapunov direct method)
Let $x = 0$ be an equilibrium point for (3.1) and let $D \subset R^n$ be a domain containing the origin. If there is a *continuously differentiable positive definite* function $V(x) : D \to R$, whose *time derivative along the system trajectories* is negative semidefinite in D

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \tag{3.6}$$

then the equilibrium is *stable*. Moreover, if $\dot{V}(x) < 0$ in $D - \{0\}$, then the equilibrium is *asymptotically stable*.

Definition 3.4
A continuously differentiable positive definite function $V(x)$ satisfying (3.6) is called a *Lyapunov function*.

Example 3.1 (prove it)
Consider the 1st order ODE

$$\dot{x} = -c(x)$$

where $c(x)$ is locally Lipschitz on $(-a,\ a)$ and satisfies

$$\{c(0) = 0\} \wedge \{c(x)x > 0, \forall\, x \neq 0 : x \in (-a \quad a)\}$$

One can show that both $V(x) = \int_0^x c(y)dy$ and $V(x) = x^2$ are the Lyapunov functions and consequently the origin is an asymptotically stable equilibrium (locally) of the system.

When the origin $x = 0$ is an asymptotically stable equilibrium of the system, we are often interested in determining its *region of attraction*, (also called region of asymptotic stability, domain of attraction, or basin). We want to be able to answer the question: Under what condition will the region of attraction be the whole space R^n ?

Definition 3.5
If the region of attraction of an asymptotically stable equilibrium point at the origin is the whole space R^n, the equilibrium is said to be *globally asymptotically stable*.

Definition 3.6
A function $V : R^n \to R$ such that $\lim_{\|x\| \to \infty} V(x) = \infty$ is called *radially unbounded*.

Theorem 3.3 (Barbashin-Krasovskii theorem)
Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a radially unbounded Lyapunov function of the system. Then the equilibrium is globally asymptotically stable.

Lecture 2

4. LaSalle's Invariance Principle

Reading material:
[1]: Chapter 3, Sections 3.4.3
[2]: Chapter 4, Section 4.2

We begin with a motivating example.

Example 4.1 (nonlinear pendulum dynamics with friction)

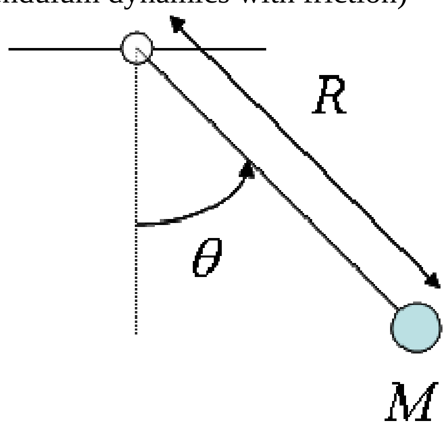

Figure 4.1: Pendulum

Dynamics of a pendulum with friction can be written as:

$$M\,R^2\,\ddot{\theta} + k\,\dot{\theta} + M\,g\,R\sin(\theta) = 0 \tag{4.1}$$

or, equivalently in state space form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - b\,x_2 \end{aligned} \tag{4.2}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, $a = \dfrac{g}{R}$, and $b = \dfrac{k}{M\,R^2}$. We study stability of the origin $x_e = 0$. Note that the latter is equivalent to studying stability of all the equilibrium points in the form: $x_e = \begin{pmatrix} 2\pi\,l & 0 \end{pmatrix}^T$, $l = 0, \pm 1, \pm 2, \ldots$ Consider the total energy of the pendulum as a Lyapunov function candidate.

$$V(x) = \underbrace{\int_0^{x_1} a \sin y \, dy}_{\text{Potential}} + \underbrace{\frac{x_2^2}{2}}_{\text{Kinetic}} = a(1 - \cos x_1) + \frac{x_2^2}{2} \tag{4.3}$$

It is clear that $V(x)$ is a positive definite function, (locally, around the origin). Its time derivative along the system trajectories is:

$$\dot{V}(x) = a \sin x_1 \, \dot{x}_1 + x_2 \, \dot{x}_2 = -b \, x_2^2 \le 0 \tag{4.4}$$

The time derivative is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1. Therefore, we can conclude that the origin is a stable equilibrium.

However, using the phase portrait of the pendulum equation (or just common sense), we expect the origin to be an asymptotically stable equilibrium. Consequently, the Lyapunov energy function argument fails to show this fact.

On the other hand, we notice that for the system to maintain $\dot{V}(x) = 0$ condition, the trajectory must be confined to the line $x_2 = 0$. Using the system dynamics (4.2) yields:

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow \sin x_1 \equiv 0 \Rightarrow x_1 \equiv 0$$

Hence on the segment $-\pi < x_1 < \pi$ of the line $x_2 = 0$ the system can ***maintain*** the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Therefore, $V(x(t))$ must decrease to toward 0 and, consequently, $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

The forgoing argument shows that if in a domain about the origin we can find a Lyapunov function whose derivative along the system trajectories is negative semidefinite, and we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$, except at the origin, then the origin is asymptotically stable. This argument follows from the *LaSalle's Invariance Principle*.

Definition 4.1
A set $M \subset R^n$ is said to be

- an *invariant set* with respect to (3.1) if: $x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in R$
- a *positively invariant set* with respect to if: $x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \ge 0$

Theorem 4.1 (LaSalle's theorem)

Let $\Omega \subset D \subset R^n$ be a *compact positively invariant set* with respect to the system dynamics (3.1). Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x(t)) \le 0$ in Ω. Let $E \subset \Omega$ be the set of all points in Ω where $\dot{V}(x) = 0$. Let $M \subset E$ be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$, that is

$$\lim_{t\to\infty}\left(\underbrace{\inf_{z\in M}\left\|x(t)-z\right\|}_{\operatorname{dist}(x(t),M)}\right)=0$$

Notice that the inclusion of the sets in the LaSalle's theorem is:

$$M \subset E \subset \Omega \subset D \subset R^n$$

In fact, the formal proof of the theorem (see [2], Theorem 4.4, p. 128) reveals that all trajectories $x(t)$ are bounded and approach a *positive limit set* $L^+ \subset M$ as $t \to \infty$. The latter may contain asymptotically stable equilibriums and stable limit cycles.

Remark 4.1
Unlike Lyapunov theorems, LaSalle's theorem *does not* require the function $V(x)$ to be positive definite.

Most often, our interest will be to show that $x(t) \to 0$ as $t \to \infty$. For that we will need to establish that the largest invariant set in E is the origin, that is: $M = \{0\}$. This is done by showing that no solution can stay identically in E other than the trivial solution $x(t) \equiv 0$.

Theorem 4.1 (Barbashin-Krasovskii theorem)
Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a *continuously differentiable positive definite* function on a domain $D \subset R^n$ containing the origin, such that $\dot{V}(x(t)) \le 0$ in D. Let $S = \{x \in D : \dot{V}(x) = 0\}$ and suppose that no other solution can stay in S, other than the trivial solution $x(t) \equiv 0$. Then the origin is *locally asymptotically stable*. If, in addition, $V(x)$ is *radially unbounded* then the origin is *globally asymptotically stable*.

Note that if $\dot{V}(x)$ is negative definite then $S = \{0\}$ and the above theorem coincides with the Lyapunov 2nd theorem. Also note that the LaSalle's invariant set theorems are applicable to autonomous system only.

Example 4.2
Consider the 1st order system

$$\dot{x} = a\,x + u$$

together with its ***adaptive control law***

$$u = -\hat{k}(t)\,x$$

The dynamics of the adaptive gain $\hat{k}(t)$ is

$$\dot{\hat{k}} = \gamma\,x^2$$

where $\gamma > 0$ is called the adaptation rate. Then the closed-loop system becomes:

$$\begin{cases} \dot{x} = -\left(\hat{k}(t) - a\right)x \\ \dot{\hat{k}} = \gamma\,x^2 \end{cases}$$

The line $x = 0$ represents the system equilibrium set. We want to show that the trajectories approach this equilibrium set, as $t \to \infty$, which means that the adaptive controller regulates $x(t)$ to zero in the presence of constant uncertainty in a. Consider the Lyapunov function candidate

$$V\left(x, \hat{k}\right) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\left(\hat{k} - b\right)$$

where $b > a$. The time derivative of V along the trajectories of the system is given by

$$\dot{V}\left(x, \hat{k}\right) = x\,\dot{x} + \frac{1}{\gamma}\left(\hat{k} - b\right)\dot{\hat{k}} = -x^2\left(\hat{k} - a\right) + \left(\hat{k} - b\right)x^2 = -x^2\left(b - a\right) \le 0$$

Since $V\left(x, \hat{k}\right)$ is positive definite and radially unbounded function, whose derivative $\dot{V}\left(x, \hat{k}\right) \le 0$ is semi-negative, the set $\Omega_c = \left\{\left(x, \hat{k}\right) \in R^2 : V\left(x, \hat{k}\right) \le c\right\}$ is compact, positively invariant set. Thus taking $\Omega = \Omega_c$, all the conditions of LaSalle's Theorem are satisfied. The set E is given by $E = \left\{\left(x, \hat{k}\right) \in \Omega_c : x = 0\right\}$. Because any point on the line $x = 0$ is an equilibrium point, E is an invariant set. Therefore, in this example $M = E$. From LaSalle's Theorem we conclude that every trajectory starting in Ω_c approaches E, as $t \to \infty$, that is $x(t) \to 0$ as $t \to \infty$. Moreover, since $V\left(x, \hat{k}\right)$ is radially unbounded, the conclusion is global, that is it holds for *all* initial conditions $\left(x(0), \hat{k}(0)\right)$ because the constant c in the definition of Ω_c can be chosen large enough that $\left(x(0), \hat{k}(0)\right) \in \Omega_c$.

Homework Assignment:

- simulate the adaptive control example using MATLAB
- test different initial conditions $\left(x(0), \hat{k}(0)\right)$
- run simulations of the system while increasing the rate of adaptation $\gamma > 0$ until high frequency oscillations and / or system departure occurs. Try to quantify maximum allowable $\gamma_{\max}$ as a function of the initial conditions.
- run simulations of the system while increasing the control time delay $\tau \geq 0$, that is using the control in the form $u(t) = -\hat{k}(t-\tau)x(t-\tau)$. Try to quantify maximum allowable time delay $\tau_{\max}$ (perhaps, as a function of the initial conditions) before the system starts to oscillate at a high frequency or simply departs.

5. Boundedness and Ultimate Boundedness

Reading material:
[2]: Section 4.8

Consider the nonautonomous system

$$\dot{x} = f(t, x) \tag{5.1}$$

where $f:[0,\infty) \times D \to R^n$ is piecewise continuous in t, locally Lipschitz in x on $[0,\infty) \times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$.

Note that if the origin is an equilibrium point for (5.1) then by definition: $f(t,0) = 0, \quad \forall t \geq 0$. On the other hand, even if there is no equilibrium at the origin, Lyapunov analysis can still be used to show boundedness of the system trajectories. We begin with a motivating example.

Example 5.1
Consider the IVP with nonautonomous scalar dynamics

$$\begin{aligned} \dot{x} &= -x + \delta \sin t \\ x(t_0) &= a > \delta > 0 \end{aligned} \tag{5.2}$$

The system has no equilibrium points. The IVP explicit solution can be easily found and shown to be bounded for all $t \geq t_0$, uniformly in t_0, that is with a bound b independent of

t_0. In this case, the solution is said to be *uniformly ultimately bounded* (UUB), and b is called the ultimate bound, (prove it).

Turns out, the UUB property of (5.2) can be established via Lyapunov analysis and without using the explicit solution of the state equation. In fact, starting with $V(x) = \frac{x^2}{2}$, we calculate the time derivative of V along the system trajectories.

$$\dot{V}(x) = x\,\dot{x} = x(-x + \delta \sin t) = -x^2 + \delta\, x \sin t \le -x^2 + \delta |x| = -|x|(|x| - \delta)$$

It immediately follows that

$$\dot{V}(x) < 0, \quad \forall |x| > \delta$$

In other words, the time derivative of V is negative outside the set $B_\delta = \{|x| \le \delta\}$, or equivalently, all solutions that start outside of B_δ will enter the interval within a finite time, and will remain within the interval bounds afterward. Formally, it can be stated as follows.

Choose $c > \frac{\delta^2}{2}$. Then all solutions starting in the set

$$B_c = \underbrace{\{V(x) \le c\}}_{|x| \le \sqrt{2c}} \supset B_\delta$$

will remain therein for all future time since $\dot{V}$ is negative on the boundary. Hence the solutions are *uniformly bounded*.

Moreover, an ultimate bound of the solutions can also be found. Choose ε such that

$$\frac{\delta^2}{2} < \varepsilon < c$$

Then $\dot{V}$ is negative in the annulus set $\{\varepsilon \le V(x) \le c\}$, which implies that in this set $V(x(t))$ will decrease monotonically in time until the solution enters the set $\{V(x) \le \varepsilon\}$. From that time on, cannot leave the set because again $\dot{V}$ is negative on its boundary $V(x) = \varepsilon$. Since $V(x) = \frac{x^2}{2}$, we can conclude that the solution is UUB with the *ultimate bound* $|x| \le \sqrt{2\varepsilon}$.

Lecture 3

Definition 5.1

The solutions of (5.1) are

- *uniformly bounded* if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall t \geq t_0 \tag{5.3}$$

- *globally uniformly bounded* if (5.3) holds for arbitrarily large a
- *uniformly ultimately bounded* with ultimate bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a,b)$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \quad \forall t \geq t_0 + T \tag{5.4}$$

- *globally uniformly ultimately bounded* if (5.4) holds for arbitrarily large a.

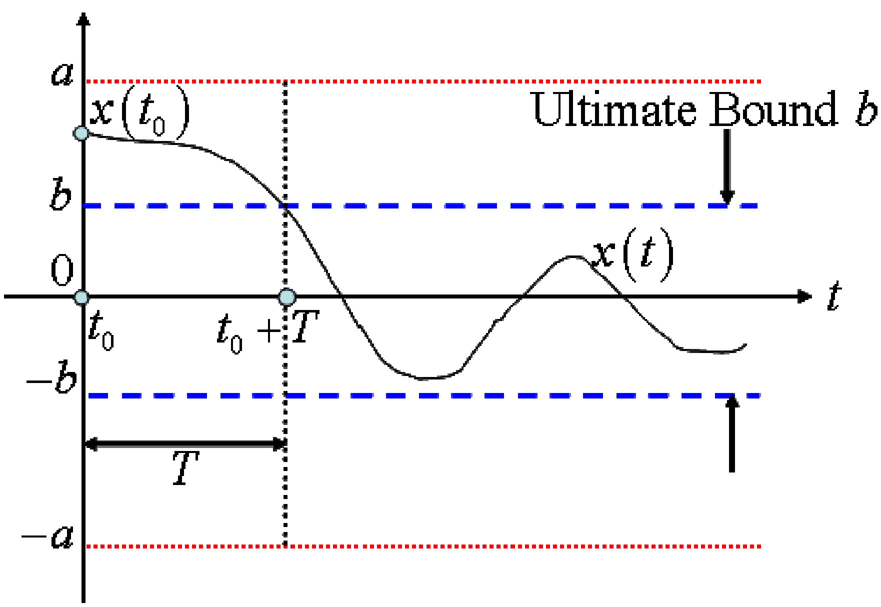

Figure 5.1: UUB Concept

In the definition above, the term *uniform* indicates that the bound b does not depend on t_0. The term *ultimate* indicates that boundedness holds after the lapse of a certain time T. The constant c defines a neighborhood of the origin, independent of t_0, such that all trajectories starting in the neighborhood will remain bounded in time. If c can be chosen arbitrarily large then the UUB notion becomes global.

Basically, UUB can be considered as a "milder" form of stability in the sense of Lyapunov (SISL). A comparison between SISL and UUB concepts is given below.

- SISL is defined with respect to an equilibrium, while UUB is not.
- Asymptotic SISL is a strong property that is very difficult to achieve in practical dynamical systems.
- SISL requires the ability to keep the state arbitrarily close to the system equilibrium by starting sufficiently close to it. This is still too strong a requirement for practical systems operating in the presence of unknown disturbances.
- The main difference between UUB and SISL is that the UUB bound b cannot be made arbitrarily small by starting closer to the equilibrium or the origin. In practical systems, the bound b depends on disturbances and system uncertainties.

To demonstrate how Lyapunov analysis can be used to study UUB, consider a continuously differentiable positive definite function $V(x)$. Choose $0 < \varepsilon < c$. Suppose that the sets $\Omega_\varepsilon = \{V(x) \le \varepsilon\}$ and $\Omega_c = \{V(x) \le c\}$ are compact. Let

$$\Lambda = \{\varepsilon \le V(x) \le c\} = \Omega_c - \Omega_\varepsilon$$

and suppose that it is known that the time derivative of $V(x(t))$ along the trajectories of the nonautonomous dynamical system (5.1) is negative definite inside Λ, that is

$$\dot{V}(x(t)) \le -W(x(t)) < 0, \quad \forall\, x \in \Lambda, \quad \forall\, t \ge t_0$$

where $W(x(t))$ is a continuous positive definite function. Since $\dot{V}$ is negative in Λ, a trajectory starting in Λ must move in the direction of decreasing $V(x(t))$. In fact, it can be shown that in the set Λ the trajectory behaves as if the origin was uniformly asymptotically stable, (which it does not have to be in this case). Consequently, the function $V(x(t))$ will continue decreasing until the trajectory enters the set Ω_ε *in finite time* and stays there for all future time. Hence, the solutions of (5.1) are UUB with the ultimate bound $b = \max\limits_{x \in \Omega_\varepsilon} \|x\|$. A sketch of the sets Λ, Ω_c, Ω_ε is shown in Figure 5.2.

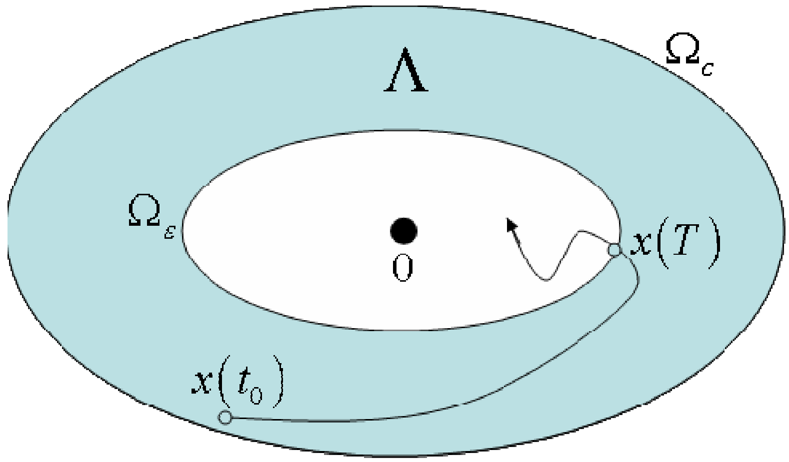

Figure 5.2: UUB by Lyapunov Analysis

In many problems, the relation $\dot{V}(t,x) \le -W(x)$ is derived and shown to be valid on a domain which is specified in terms of $\|x\|$. In such cases, the UUB analysis involves finding the corresponding domains of attraction and an ultimate bound. In particular, suppose that

$$V(x) = x^T P x \tag{5.5}$$

where $P = P^T > 0$ is a positive definite symmetric matrix. Also suppose that the time derivative of V evaluated along the system (5.1) trajectories satisfies the following relation:

$$\dot{V}(x) = -x^T Q x + x^T P b \varepsilon(t,x), \quad \forall x \in S_R \triangleq \{\|x\| \le R\}, \quad \forall t \ge t_0 \tag{5.6}$$

where $R > 0$ is the radius of the sphere S_R, $Q = Q^T > 0$, $b \in R^n$ is a constant known vector, and $|\varepsilon(t,x)| \le \varepsilon_{\max} < \infty$ for all $x \in S_R$, uniformly in t. Then it can be shown that all the solutions $x(t)$ of (5.1) are UUB. The *formal proof* goes as follows.

We start with the well-known double-inequality, which is valid for any positive definite matrix P and for all vectors x.

$$\lambda_{\min}(P)\|x\|^2 \le x^T P x \le \lambda_{\max}(P)\|x\|^2 \tag{5.7}$$

where $\lambda_{\min}(P), \lambda_{\max}(P)$ denote the smallest and the largest eigenvalues of P, respectively.

An upper bound for $\dot{V}$ in (5.6) can be calculated.

$$\dot{V}(x) \le -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|Pb\|\varepsilon_{\max} = -\|x\|\left(\lambda_{\min}(Q)\|x\| - \|Pb\|\varepsilon_{\max}\right) \tag{5.8}$$

Let $S_r \triangleq \left\{ \|x\| \le r \triangleq \frac{\|Pb\|\varepsilon_{\max}}{\lambda_{\min}(Q)} \right\}$. Then it follows from (5.8) that

$$\dot{V}(x) < 0, \quad \forall x \in \Lambda = \left\{ r \le \|x\| \le R \right\} = S_R - S_r \tag{5.9}$$

Let $b = \lambda_{\max}(P) r^2$ and define $\Omega_b \triangleq \left\{ V(x) \le b \right\}$. Then $S_r \subset \Omega_b$. In fact, if $x \in S_r$ then using the right hand side of (5.7) yields:

$$x^T P x \le \lambda_{\max}(P)\|x\|^2 \le \lambda_{\max}(P) r^2 = b \tag{5.10}$$

Hence, $x \in \Omega_b$ and the inclusion $S_r \subset \Omega_b$ is proven.

Let $B = \lambda_{\min}(P) R^2$ and define $\Omega_B \triangleq \left\{ V(x) \le B \right\}$. Then $\Omega_B \subset S_R$. In fact, if $x \in \Omega_B$ then using the left hand side of (5.7) yields:

$$\lambda_{\min}(P)\|x\|^2 \le x^T P x \le B = \lambda_{\min}(P) R^2 \tag{5.11}$$

Hence, $\|x\| \le R$, that is $x \in S_R$, and the inclusion $\Omega_B \subset S_R$ is proven.

Next we need to make sure that $b < B$. The latter implies

$$b = \lambda_{\max}(P) r^2 < \lambda_{\min}(P) R^2 = B \tag{5.12}$$

or, equivalently:

$$\frac{r}{R} < \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \tag{5.13}$$

The above relation can be viewed as a restriction on the eigenvalues of P and the constants r and R. Basically, inequality (5.13) ensures that

$$S_r \subset \Omega_b \subset \Omega_B \subset S_R \tag{5.14}$$

Graphical representation of the four sets is given in Figure 5.2.

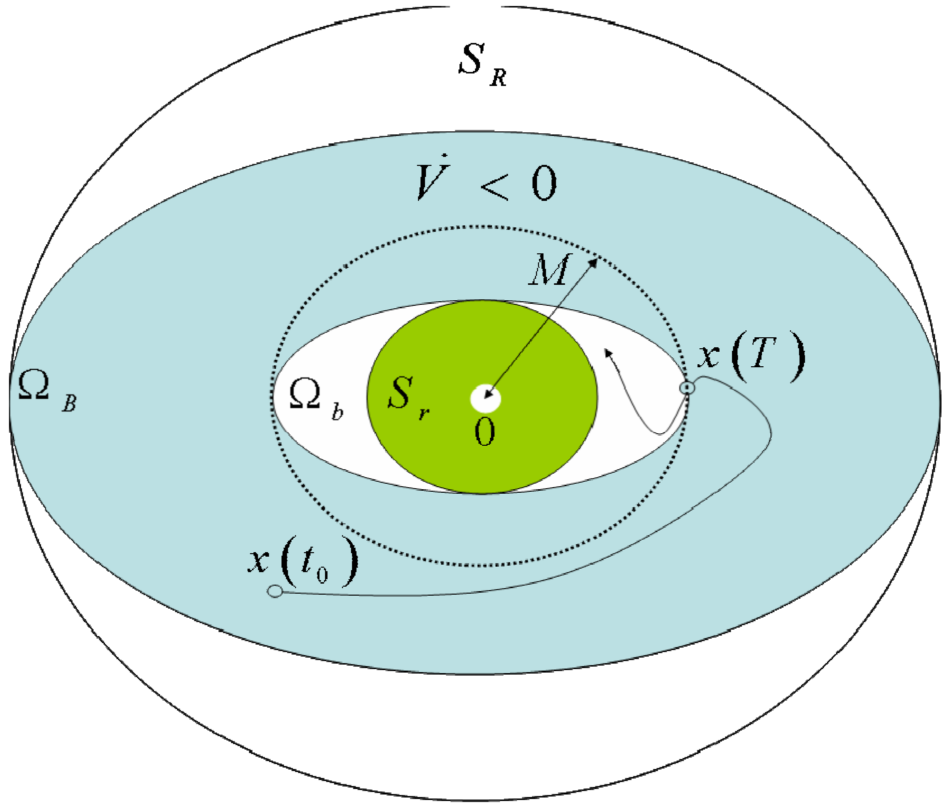

Figure 5.2: Representation of the sets $S_r \subset \Omega_b \subset \Omega_B \subset S_R$ and the Ultimate Bound M

Next we show that all solutions starting in Ω_B will enter Ω_b and remain there afterwards.

If $x(t_0) \in \Omega_b$ then since $\dot{V} < 0$ in $\Lambda = \Omega_B - \Omega_b$, $V(x(t))$ is a decreasing function of time outside of Ω_b. Therefore, solutions that start in Ω_b will remain there.

Suppose that $x(t_0) \in \Lambda$. Inequality (5.8) implies that

$$\begin{aligned}
&\dot{V}(x) \le -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|Pb\|\varepsilon_{\max} \\
&= -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \underbrace{\lambda_{\max}(P)\|x\|^2}_{\ge V(x)} + \frac{\|Pb\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}} \underbrace{\sqrt{\lambda_{\min}(P)}\|x\|}_{\le \sqrt{V(x)}} \\
&\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(x) + \frac{\|Pb\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}} \sqrt{V(x)}
\end{aligned} \tag{5.15}$$

Thus, $V(x(t)) \ge 0$ satisfies the following differential inequality, (as a function of time):

$$\dot{V}(x) \le -aV(x) + g\sqrt{V(x)} \tag{5.16}$$

where $a = \dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$ and $g = \dfrac{\|Pb\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}$ are positive constants. Let $W(x) = \sqrt{V(x)}$. Then relation (5.16) is equivalent to

$$\dot{W}(x) \le -\frac{a}{2}W(x) + \frac{g}{2} \tag{5.17}$$

Define

$$Z(x) \triangleq \dot{W}(x) + \frac{a}{2}W(x) \tag{5.18}$$

Inequality (5.17) implies that

$$Z(x(t)) \le \frac{g}{2}, \quad \forall t \ge t_0 \tag{5.19}$$

Solving (5.18) for W yields:

$$W(x(t)) = W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} Z(x(\tau))\,d\tau \tag{5.20}$$

Therefore

$$\begin{aligned} W(x(t)) &\le W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} \left|Z(x(\tau))\right| d\tau \\ &\le W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \frac{g}{2}\int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)}\,d\tau = W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \frac{g}{a}\left[1 - e^{-\frac{a}{2}(t-t_0)}\right] \\ &= \frac{g}{a} + \underbrace{e^{-\frac{a}{2}(t-t_0)}\left[W(x(t_0)) - \frac{g}{a}\right]}_{\mathrm{o}(1)} = \frac{g}{a} + \mathrm{o}(1) \end{aligned} \tag{5.21}$$

As $t \to \infty$ and in terms of the original variables, one gets

$$\sqrt{x^T(t)P\,x(t)} \le \frac{g}{a} + \mathrm{o}(1) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\underbrace{\left(\frac{\|Pb\|\varepsilon_{\max}}{\lambda_{\min}(Q)}\right)}_{r} + \mathrm{o}(1) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\,r + \mathrm{o}(1) \tag{5.22}$$

Choose $\delta > 0$. Then it is easy to see that there exists T independent of t_0 such that $\mathrm{o}(1) \le \delta$ and , consequently

$$\sqrt{x^T(t)\,P\,x(t)} \le \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\,r + \delta, \quad \forall\, t \ge T + t_0, \quad \forall\, x \in \Omega_B \tag{5.23}$$

Since the above relation is valid for all solutions that start in Λ, it is also valid for the solution which starts in Λ and maximizes the left hand side of the inequality. In other words

$$\max_{x\in\Omega_B} \sqrt{x^T(t)\,P\,x(t)} \le \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\,r + \delta \tag{5.24}$$

On the other hand

$$\max_{x\in\Omega_B} \sqrt{x^T(t)\,P\,x(t)} = \sqrt{\lambda_{\max}(P)}\,\max_{x\in\Omega_B}\left\|x(t)\right\| \tag{5.25}$$

Hence

$$\left\|x(t)\right\| \le M \triangleq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,r + \frac{\delta}{\sqrt{\lambda_{\max}(P)}}, \quad \forall\, t \ge T + t_0, \quad \forall\, x \in \Omega_B \tag{5.26}$$

Consequently, the solutions of (5.1) are UUB with the ultimate bound M. The bound is shown in Figure 5.2. Summarizing the results, we state the following theorem.

Theorem 5.1
Let $P = P^T > 0$ and $Q = Q^T > 0$ be symmetric positive definite matrices and let $V(x) = x^T P\,x$. Suppose that the time derivative of V along the trajectories of (5.1) satisfies (5.6), for all $x \in S_R = \left\{\|x\| \le R\right\}$. Let $S_r \triangleq \left\{\|x\| \le r \triangleq \frac{\|P\,b\|\,\varepsilon_{\max}}{\lambda_{\min}(Q)}\right\}$ and suppose that $\frac{r}{R} < \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$. Then the solutions are UUB with the ultimate bound M in (5.26).

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