Introduction to Chaos and Symbol Dynamics

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1 Introduction to Chaos.

The Study of Deterministic Chaos. *Despite the fact that the system is deterministic, it has the property that imprecise knowledge of the intial condition may lead to unpredictability after some finite time.*

Example: Look at the phase space of a pendulum and a pendulum with perodical forcing. Notice the complexity of the homoclinic tangle.

Overview: We will cover

- **Symbolic Dynamics** which is the paradigm for deterministic chaos.
- **Conley-Moser Conditions** which allow one to verify the existence of **Smale Horseshoe**-like dynamics and chaos.
- **Homoclinic Orbits and Heteroclinic Cycles** where horseshoe-like dynamics exists and where the whole machinery of symbolic dynamics can make this chaotic behavior more precise.

2 Symbolic Dynamics and the Shift Map

Phase Space. The phase space for the shift map Σ is the space of "bi-infinite" sequences of 0's and 1's, with a specific metric. Two sequences are "near each other" if they are identical on a long central block.

Σ is a Cantor Set. It can be proved that Σ is compact, perfect (i.e., every point is a limit point), and totally disconnected, i.e., Σ is a Cantor set.

Example: Classical Cantor "Middle-Thirds" Set.

The Shift Map σ on Σ. σ is a homeomorphism and that it has the following properties:

1. a countable ∞ of periodic orbits of all periods,
2. an uncountable ∞ of nonperiodic orbits, and
3. a "dense orbit", i.e., an orbit that is dense in Σ.

Deterministic Chaos. The dynamics of $\sigma : \Sigma \to \Sigma$ model the phenomenon of deterministic chaos which has the following ingredients:

- Σ is compact;
- σ is topologically transitive, i.e., given any two open sets in Σ some iterate of one will intersect the other (this essentially follows from the existence of the dense orbit);
- σ exhibits sensitive dependence on initial conditions, i.e., the distance between nearby initial conditions grows under some fixed number of iterates.

Remark: Despite the fact that the system is deterministic, it has the property that imprecise knowledge of the intial condition may lead to unpredictability after some finite time.