

Lecture 1B: Introduction to Perturbation Theory

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1 Basic Ideas

- **Perturbed and unperturbed problems:** To find approximated solutions to a perturbed problem, one will always assume that one has sufficient knowledge of the solutions of the unperturbed problem.
- **Time scale:** In constructing approximated solutions, one has to indicate on what interval of time (time-scale) one is looking for an approximation.

Example (Duffing equation):

$$\ddot{q} + q + \epsilon q^3 = 0.$$

2 Naive Expansion

Theorem 9.1: Consider the initial value problem

$$\dot{x} = f_0(t, x) + \epsilon f_1(t, x) + \dots + \epsilon^m f_m(t, x) + \epsilon^{m+1} R(t, x, \epsilon)$$

with $x(t_0) = \eta$ and $|t - t_0| \leq h, x \in D \subset \mathbb{R}^n, 0 \leq \epsilon \leq \epsilon_0$. Assume that in this domain

1. $f_i(t, x), i = 0, \dots, m$ continuous in t and x , $(m + 1 - i)$ times continuously differentiable in x ;
2. $R(t, x, \epsilon)$ continuous in t, x and ϵ , Lipschitz-continuous in x .

Substituting in the equation for x the formal expansion

$$x_0(t) + \epsilon x_1(t) + \dots + \epsilon^m x_m(t),$$

Taylor expanding w.r.t. powers of ϵ , equating corresponding coefficients and applying the initial values $x_0(t_0) = \eta, x_i(t_0) = 0, i = 1, \dots, m$ produces an approximation of $x(t)$:

$$\|x(t) - (x_0(t) + \epsilon x_1(t) + \dots + \epsilon^m x_m(t))\| = O(\epsilon^{m+1})$$

on the time-scale 1.

3 The Poincaré Expansion Theorem

Theorem 9.2 (Poincaré Expansion Theorem) Consider the initial value problem

$$\dot{y} = F(t, y, \epsilon), \quad y(t_0) = \mu,$$

with $|t - t_0| \leq h, y \in D \subset R^n, 0 \leq \epsilon \leq \epsilon_0, 0 \leq \mu \leq \mu_0$. If $F(t, y, \epsilon)$ is continuous w.r.t. t, y and ϵ and can be expanded in a convergent power series w.r.t. y and ϵ for $\|y\| \leq \rho, 0 \leq \epsilon \leq \epsilon_0$, then $y(t)$ can be expanded in a convergent power series w.r.t. ϵ and μ in a neighborhood of $\epsilon = \mu = 0$, convergent on the time-scale 1.

Remark: Poincaré Expansion Theorem is a part of the preparation to find convergent series approximation of periodic solutions.