Smale Horseshoe and Conley-Moser Conditions

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1 Introduction to Chaos.

The Study of Deterministic Chaos. *Despite the fact that the system is deterministic, it has the property that imprecise knowledge of the intial condition may lead to unpredictability after some finite time.* We will cover

- **Symbolic Dynamics** which is the paradigm for deterministic chaos.
- **Conley-Moser Conditions** which allow one to verify the existence of **Smale Horseshoe**-like dynamics and chaos.
- **Homoclinic Orbits and Heteroclinic Cycles** where horseshoe-like dynamics exists and where the whole machinery of symbolic dynamics can make this chaotic behavior more precise.

2 Symbolic Dynamics and the Shift Map

Phase Space. The phase space for the shift map Σ is the space of "bi-infinite" sequences of 0's and 1's, with a specific metric. Two sequences are "near each other" if they are identical on a long central block.

The Shift Map σ on Σ. σ is a homeomorphism and that it has the following properties:

1. a countable ∞ of periodic orbits of all periods,
2. an uncountable ∞ of nonperiodic orbits, and
3. a "dense orbit", i.e., an orbit that is dense in Σ.

Deterministic Chaos. The dynamics of $\sigma : \Sigma \to \Sigma$ model the phenomenon of deterministic chaos which has the following ingredients:

- Σ is compact;
- σ is topologically transitive, i.e., given any two open sets in Σ some iterate of one will intersect the other (this essentially follows from the existence of the dense orbit);
- σ exhibits sensitive dependence on initial conditions, i.e., the distance between nearby initial conditions grows under some fixed number of iterates.

3 Smale Horseshoe

Horseshoe Map is a two-dimensional map

$$f : D \to D$$

where f contracts the square D in the horizontal direction, expand it in the vertical direction, and folds it onto itself in such a way that

- the two "horizontal" rectangles H_0 and H_1 are mapped to the two vertical rectangle V_0 and V_1 respectively;
- horizontal boundaries of H_i map to horizontal boundaries of V_i, and vertical boundaries of H_i map to veritical boundaries of V_i.

Invariant Set of Horseshoe Map. Consider the invariant set of this map

$$\Lambda = \cap_{-\infty}^{\infty} f^n(D).$$

Λ is the set of points in the square that remain in the square under all forward and backward iterations by f.

Cantor Set of Vertical Lines. Notice that $f(D \cap f(D)) \cap D$ consists of four vertical rectangles. This process can be carried out indefinitely. Passing to the limit as $n \to \infty$ gives

$$\Lambda_- = D \cap f(D) \cap \ldots \cap f^n(D) \ldots,$$

which is a Cantor set of vertical lines. These vertical lines are the sets of points in D that remain in D under all *backward* iterations of f.

Cantor Set of Horizontal Lines. Similarly, $f^{-1}(D \cap f^{-1}(D)) \cap D$ consists of four horizontal rectangles. Passing to the limit as $n \to \infty$ gives

$$\Lambda_+ = D \cap f^{-1}(D) \cap \ldots \cap f^{-n}(D) \ldots,$$

which is a Cantor set of horizontal lines, which are the set of points n D that remain in D under all *forward* iterations of f.

Cantor Set of Points. Hence the set

$$\Lambda = \Lambda_+ \cap \Lambda_-$$

is a Cantor set of points. These are the points in D that remain in D under all iterations by f.

Dynamics on the Invariant Set. The dynamics on Λ of f is "modelled" by the shift map σ. There is a change of coordiates that transform f, restricted to Λ, to $\sigma : \Sigma \to \Sigma$.

Horseshoe Dynamics is Chaotic. Every orbit of the shift map is mapped uniquely to an orbit of f_Λ. f_Λ has the same orbit structure as σ:

- it has a countable infinity of periodic orbits of all periods,
- an uncountable inifinity of non-periodic orbits, and
- an orbit that is dense in Λ.

It is also chaotic in the same sense as the shift map.

4 Conley-Moser Conditions

Conley-Moser Conditions for a general two-dimensional map to have an invariant set on which it has the shift dynamics. These conditions are a combination of geometric and analytical conditions.

- The geometrical part consists of generalizing the notion of horizontal and vertical rectangles by allowing the boundaries to be Lipschitz curves, rather than straight lines.
  - horizontal "rectangles" map to "vertical" rectangles,
  - with horizontal boundaries map to horizontal boundaries and vertical boundaries map to vertical boundaries.
- The analytical part requires uniform (but not constant) contraction in the horizontal directions and expansion in the vertical direction.