Hopf Bifurcation

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1 Introduction

Consider

$$\dot{w} = g(w, \eta).$$

where $w \in R^n, \eta \in R^p$. Suppose it has a fixed point at (w_0, η_0), i.e., $g(w_0, \eta_0) = 0$. Moreover, its linearized equation

$$\dot{\xi} = D_w g(w_0, \eta_0)\xi$$

has two purely imaginary eigenvalues with the remaining $(n-2)$ eigenvalues having nonzero real parts.

Center Manifold Theorem tells us that the orbit structure near (w_0, η_0) is determined by the associated equation on the two-dimensional center manifold, which can be written in the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha(\mu) & -\omega(\mu) \\ \omega(\mu) & \alpha(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, \mu) \\ f^2(x, y, \mu) \end{pmatrix}$$

where $\lambda(\mu) = \alpha(\mu) \pm i\omega(\mu)$ (with $\lambda(0) = \pm i\omega(0)$ are the eigenvalues of the linearized equation.

Normal Form was found to be

$$\begin{aligned} \dot{r} &= \alpha(\mu)r + a(\mu)r^3 + O(r^5), \\ \dot{\theta} &= \omega(\mu) + b(\mu)r^2 + O(r^4). \end{aligned}$$

Remark: Notice that

$$d = \frac{d}{d\mu}(\mathrm{Re}\lambda(\mu))|_{\mu=0}.$$

Hence, for $d > 0$, the eigenvalues cross from the left half-plane to the right half-plane as μincreases and, for $d < 0$, the eigenvalues cross from the right half-plane to the left half-plane as μ increases.

Two Steps.

1. Neglect the higher order terms and study the resulting "truncated" normal form.

2. Show that the dynamics are qualatively unchanged when higher order terms are considered.

2 Firt Step

Consider

$$\begin{aligned} \dot{r} &= d\mu r + ar^3, \\ \dot{\theta} &= \omega + c\mu + br^2. \end{aligned}$$

Lemma 3.2.1 For $-\infty < \frac{\mu d}{a} < 0$ and μ sufficiently small

$$(r(t), \theta(t)) = \left(\sqrt{\frac{-\mu d}{a}}, \left[\omega + \left(c - \frac{bd}{a} \right) \mu \right] t + \theta_0 \right)$$

is a periodic orbit.

Lemma 3.2.2 The periodic orbit is

1. asymptotically stable for $a < 0$;

2. unstable for $a > 0$

Case 1: $d > 0, a > 0$**.**

Case 2: $d > 0, a < 0$**.**

Case 3: $d < 0, a > 0$**.**

Case 4: $d < 0, a < 0$**.**

Remark: Supercritical and Subcritical bifurcation.

3 Second Step

Poincaré-Andronov-Hopf Bifurcation Consider the full normal form. Then, for μ sufficiently small, cases 1-4 hold.

Example: Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu + 2 & -5 \\ 1 & \mu - 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (x^2 - 4xy + 5y^2)x \\ (x^2 - 4xy + 5y^2)y \end{pmatrix}$$