Chaotic Dynamics Near Transverse Homoclinic Points

CDS140B Lecturer: Wang Sang Koon

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The Study of Deterministic Chaos. *Despite the fact that the system is deterministic, it has the property that imprecise knowledge of the intial condition may lead to unpredictability after some finite time.* We will cover

- **Symbolic Dynamics** which is the paradigm for deterministic chaos.
- **Conley-Moser Conditions** which allow one to verify the existence of **Smale Horseshoe**-like dynamics and chaos.
- **Homoclinic Orbits and Heteroclinic Cycles** where horseshoe-like dynamics exists and where the whole machinery of symbolic dynamics can make this chaotic behavior more precise.

Conley-Moser Conditions for a general two-dimensional map to have an invariant set on which it has the shift dynamics. These conditions are a combination of geometric and analytical conditions.

- The geometrical part consists of generalizing the notion of horizontal and vertical rectangles by allowing the boundaries to be Lipschitz curves, rather than straight lines.
  - horizontal "rectangles" map to "vertical" rectangles,
  - with horizontal boundaries map to horizontal boundaries and vertical boundaries map to vertical boundaries.
- The analytical part requires uniform (but not constant) contraction in the horizontal directions and expansion in the vertical direction.

Goal of this Lecture is to show the following result.

Theorem (Smale [1962], Moser [1972]) In a neighborhood of a transverse homoclinic point there exists an invariant Cantor set on which the dynamics is topologically conjugate to a full shift on N symbols.

Notes: Smale formulated an provided the first proof. But the presentation here follows the work of Conley-Moser (the Conley-Moser conditions are satisfied near a transverse homoclinic point).

1 Transverse Homoclinic Points

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. Suppose

- f has a hyperbolic periodic orbit, i.e., there exists $p \in \mathbb{R}^2$ such that $f^k(p) = p$. W.O.L.G., we can assume $k = 1$.
- p is a saddle point and its stable and unstable manifold $W^s(p), W^u(p)$ intersect transversely in a point q, then q is called a *transverse homoclinic point*.

2 Smale and Moser Theorem

Proof of the theorem:

Step 1: Local Coordinates for f.

Step 2: Global Consequences.

Step 3: Dynamics Near the Origin and Remark on Lambda Lemma.

Step 4: Dynamics Away from the Origin.

Step 5: Proof of Theorem.

3 Melinkov Methods for the Existence of Homoclinic Orbits.