Introduction to Hamiltonian Systems

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winter, 2005

1 Introduction

Consider a C^2 function of $2n$ variables q_i, p_i $H : R^{2n} \to R$, then the Hamiltonian equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

It can be written in a compact form

$$\dot{z} = J\nabla H(z), \qquad z = (q, p).$$

where J is the canonical sympletic matrix. What follow are some basic facts.

1.1 Symplectic Matrix and Canonical Transformation.

A matrix M is symplectic if $MJM^T = J$. If the Jacobian of a transformation $((q,p) \to (Q,P))$ is symplectic, then it is a canonical transformation which preserves the Hamiltonian form of the equations.

1.2 $H(q,p)$ is a first integral of the equations.

1.3 Stability of Equilibria.

Nondegenerate Critical Point. Suppose $F : \mathbb{R}^n \leftarrow \mathbb{R}$ is a $C^r(r \geq 3)$ function. Supppose $x = x_0$ is a point such that $\frac{\partial F}{\partial x}(x_0) = 0$. Then x_0 is said to be a critical point. If x_0 is such that $|\frac{\partial^2 F}{\partial x^2}(x_0)| \neq 0$ then it is said to be a nondegerate critical point.

Theorem. If λ is an eigenvalue of a linearized Hamiltonian system, so are $-\lambda, \bar{\lambda}$ and $-\bar{\lambda}$.

Remark: Hamiltonian system cannot be linear stable.

Morse Lemma If F has a nondegenerate critical point at $x = 0$, then in a neighborhood of $x = 0$ there exist a C^{r-2} diffeomorphism, which transforms F to the form

$$G(y) = G(0) - y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2.$$

Theorem. If $H(q, p) - H(0, 0)$ is sign definite in a neighborhood of the critical point $(0, 0)$, the equilibrium solution $(q, p) = (0, 0)$ is stable.

Lemma: Consider $\dot{x} = f(x), x \in \mathbb{R}^n$, and suppose that it generates a flow $\phi_t(\cdot)$. Let D_0 denote a domain in $\mathbb{R}^n$ and let $D_t = \phi_t(D_0)$ denote the evolution of D_0 under the flow. Let $V(t)$ denote the volume of D_t. Then

$$\frac{dV}{dt}|_{t=0} = \int_{D_0} \nabla \cdot f dx$$

where $\nabla \cdot f$ denotes the divergence of a vector field.

Liouville's Theorem. The flow generated by a time-independent Hamiltonian system is volume preserving,

Remark: Attraction by an equilibrium solution is impossible as the flow in phase space is volume preserving (Liouville).

Poincaré Recurrence Theorem: Consider a bounded domain $D \subset \mathbb{R}^n$, g is injective, continuous, volumn-preserving mapping of D into itself. Then each neighborhood U of each point in D contains a point x which returns to U after repeated applications of the mapping ($g^n(x) \in U$ for some $n \in \mathbb{N}$).

2 Harmonic Oscillator with Two Degrees of Freedom

Consider linear harmonic oscillator with two degrees of freedom with Hamiltonian

$$H = \frac{1}{2}\omega_1(q_1^2 + p_1^2) + \frac{1}{2}\omega_2(q_2^2 + p_2^2).$$

with ω_i positive.

- Phase space is 4-dimensional. The constant energy surface ($H = E_0$) is topologically a 3-dimensional sphere S^3.
- Besides H, there are two other integrals

  $$\tau_1 = \frac{1}{2}(q_1^2 + p_1^2), \qquad \tau_2 = \frac{1}{2}(q_2^2 + p_2^2).$$

  However, there are only two functionally independent integrals as

  $$E_0 = \omega_1\tau_1 + \omega_2\tau_2.$$

- The existence of two independent integrals enables us to envisage the structure of the phase space (a foliation of invariant tori around one of the normal mode linked with a foliation of invariant tori around the other normal mode).

3 A Nonlinear Example with Two Degrees of Freedom

Consider

$$H = \frac{1}{2}(p_1^2 + 4q_1^2) + \frac{1}{2}(p_2^2 + q_2^2) - q_1 q_2^2.$$

- Origin of phase space is a stable critical point.

Averaging Method. After applying averaging method, we obtain:

Averaged System Has Two Integrals. The two integrals are:

$$\begin{aligned} 4\rho_1^2 + \rho_2^2 &= 2E_0, \\ \frac{1}{2}\rho_1\rho_2^2\cos(\psi_1 - 2\psi_2) &= I. \end{aligned}$$

Two New Families of Periodic Orbits. Besides the family of normal modes in the p_1, q_1-direction, there are two new families of periodic solutions.

Poincaré Section for the System. See the figure for the Poicaré section in the book.

4 A Few Remarks on Birkhoff-Gustavson Normal Form

Theorem for Birkhoff Normal Form: Consider Hamiltonian $H(q,p) = H_2 + H_3 + H_4 + \ldots$ and suppose the frequencies ω_i do not satisfy a resonance relation of order $\leq k$, then there exists a canonical transformation such that the new Hamiltonian is in Birkhoff normal form to degree k.

Definition: $H(q,p)$ is in Birkhoff normal form to degree k if H can be written as $H = H_k(I) + R_k$ where H_k is a a polynomial of degree k when written out in q, p variables ($I_i = \frac{1}{2}(q_i^2 + p_i^2)$) and R_k represents the higher order terms.