Center Manifold Theory

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Introduction to Bifurcation Thoery. In this chapter, we will cover the following materials:

- **Center Manifold Theory** allows us to reduce the dimension of a problem, you will most likely still be left with a nonlinear system.
- **Normal Form Theory** can be used to "simplify" the nonlinear system by (removing as much nonlinearity as possible. This involves nonlinear coordinate transformation.
- **Local Bifurcation Theory** uses the above techniques to determine when the system changes qualitatively as parameters are varied.

1 Center Manifold Theory

1.1 Existence

Theorem 13.3 (Existence). Consider

$$\dot{x} = Ax + f(x)$$

where

1. $x \in R^n$ and A is a constant $n \times n$ matrix; $x = 0$ is an isolated critical point; the vector function $f(x)$ is $C^k, k \geq 2$, in a neighborhood of $x = 0$ and $\lim_{||x|| \to 0} \; ||f(x)||/||x|| = 0$;
2. the stable and unstable manifolds of equation

$$\dot{y} = Ay$$

are E_s and E_u, the space of eigenvectors corresponding with eigenvalues with zero real part is E_c.

Then there exists a C^{k-1} invariant manifold W_c, the center manifold, which is tangent to E_c near $x = 0$; if $k = \infty$, then W_c is in general C^m with $m \leq \infty$.

Example 13.6

$$\begin{aligned}\dot{x} &= -x + y^2 \\ \dot{y} &= -y^3 + x^2\end{aligned}$$

Example 13.7 (W_c is not unique).

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= -y\end{aligned}$$

1.2 Stability

Theorem 13.4 (Stability) Consider equation (13.19)

$$\begin{aligned}\dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y)\end{aligned}$$

where A has only eigenvalues with zero real part and B has only eigenvalues with negative real part; f and g have a Taylor expansion near $(0, 0)$. Then the flow in the center manifold is determined by the following equation (13.20)

$$\dot{u} = Au + f(u, h(u)).$$

where $y = h(x)$ represents the center manifold of equation (13.19) near the isolated critical point $(0, 0)$. If the solution $u = 0$ of the equation 13.20 is stable (unstable), then the solution near $(0, 0)$ of equation 13.19 is stable (unstable).

1.3 Approximation

The center manifold $h(x)$ can be approximated by substituting a Taylor expansion into the following PDE:

$$\frac{\partial h}{\partial x}(Ax + f(x,h)) - Bh - g(x,h) = 0.$$

Example 1.1.1

$$\begin{aligned} \dot{x} &= x^2 y - x^5 \\ \dot{y} &= -y + x^2 \end{aligned}$$

Remark: The failure of the tangent space approximation.

Example 1.1.2

$$\begin{aligned}\dot{x} &= -xy - x^6\\ \dot{y} &= -y + x^2\end{aligned}$$

1.4 Center Manifolds Depending on Parameters

Example 13.8

$$\begin{aligned}\dot{x} &= \mu x - x^3 + xy\\ \dot{y} &= -y + y^2 - x^2.\end{aligned}$$

The Lorenz Equations.

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= \bar{\rho}x + x - y - xz,\\ \dot{z} &= -\beta z + xy.\end{aligned}$$