# Bifurcation of Fixed Points

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## 1 Introduction

Consider

$$\dot{y} = g(y, \lambda).$$

where $y \in R^n, \lambda \in R^p$. Suppose it has a fixed point at $(y_0, \lambda_0)$, i.e., $g(y_0, \lambda_0) = 0$.

**Two Questions:** (1) Is the fixed point stable or unstable? (2) How is the stability or instability affected as $\lambda$ is varied?

**Hyperbolic Fixed Points.** None of the eigenvalues of $D_y g(y_0, \lambda_0)$ lie on the the imaginary axis.

- The stability of $(y_0, \lambda_0)$ is determined by its linearized equation.
- Since hyperbolic fixed points are structurally stable, varying $\lambda$ slightly does not change the nature of the stability of the fixed point.

**Non-hyperbolic Fixed Points.** $D_y g(y_0, \lambda_0)$ has some eigenvalues on the imaginary axis.

- For $\lambda$ very close to $\lambda_0$ (and for $y$ very close to $y_0$), radically new dynamical behavior can occur: fixed points can be created or destroyed, and periodic, quasiperiodic, or even chaotic dynamics can be created.
- We will begin by studying the case where the linearized equation has a single zero eigenvalue with the remaining eigenvalues having nonzero real parts.

### 1.1 A Zero Eigenvalue

In this case, the orbit structure near $(y_0, \lambda_0)$ is determined by the associated center manifold equation

$$\dot{x} = f(x, \mu)$$

where $x \in R^1, \mu \in R^p$ and $\mu = \lambda - \lambda_0$. Moreover, $f(0,0) = 0$ (fixed point condition) and $D_x f(0,0) = 0$ (zero eigenvalue condition).

**Saddle-Node Bifurcation.** Consider

$$\dot{x} = \mu - x^2.$$

where $(x, \mu) = (0, 0)$ is a bifurcation point and $\mu = 0$ is a bifurcation value. $\dot{\xi} = -2x\xi$ is its linearized equation. The set of fixed points is given by $\mu = x^2$. See the bifurcation diagram below.

**Transcritical Bifurcation.** Consider

$$\dot{x} = \mu x - x^2.$$

The fixed points are given by $x = 0, x = \mu$. $\dot{\xi} = (\mu - 2x)\xi$ is its linearized equation.

**Pitchfork Bifurcation.** Consider

$$\dot{x} = \mu x - x^3.$$

The fixed points are given by $x = 0, x^2 = \mu$. $\dot{\xi} = (\mu - 3x^2)\xi$ is its linearized equation.

**No Bifurcation.** Conider

$$\dot{x} = \mu - x^3.$$

The fixed points are given by $\mu = x^3$. Notice that a fixed point is nonhyperbolic is a necessary but not sufficient condition for bifurcation to occur in an one-paramter family of vector fields.

## 1.2 Saddle-Node Bifurcation

In this case, a unique curve of fixed point $\mu(x)$ passing through $(0,0)$: (1) it is tangent to $\mu = 0$ at $x = 0$ ($\frac{d\mu}{dx}(0) = 0$); and (2) it lay entirely to one side of $\mu = 0$ ($\frac{d^2\mu}{dx^2}(0) \neq 0$).

**Definition:** A fixed point $(x, \mu) - (0,0)$ of an one-parameter family of one-dimensional systems is said to undergo a bifurcation at $\mu = 0$ if the flow for $\mu$ near zero and $x$ near zero is not qualitatively the same as the flow near $x = 0$ at $\mu = 0$.

**Theorem on Saddle-Node Bifurcation.** In order for

$$\dot{x} = f(x, \mu)$$

to undergo a saddle-node bifurcation, we must have $f(0,0) = 0; D_x f(0,0) = 0$ (nonhyperbolic fixed point), $D_\mu f(0,0) \neq 0$ (unique curve of fixed point through origin), and $D^2_{xx}(0,0) \neq 0$ (curves lies locally on one side of $\mu = 0$).

**Remark:** consider a general one-parameter family of one-dimensional vector fields having a non-hyperbolic fixed point at $(x, \mu) = (0, 0)$. The Taylor expansion is given by

$$f(x, \mu) = a_0\mu + a_1 x^2 + a_2 \mu x + a_3 \mu^2 + O(3).$$

The theorem shows that the dynamics of our equation near $(0,0)$ are qualitiatively the same as

$$\dot{x} = \mu \pm x^2$$

which can be viewed as the normal form for saddle-node bifurcation. Moreover, in this case, all terms of $O(3)$ and higher could be neglected and the dynamics would not qualitatively changed (thanks to the implicit function theorem),

### 1.3 The Transcritical Bifurcation

In this case, (1) two curves of fixed points pass through $(0,0)$, one given by $x = 0$, the other by $\mu(x)$; (2) both curves of fixed points existed on both sides of $\mu = 0$; and (3) the stability along each curve of fixed points changed on passing through $\mu = 0$.

**Theorem on Transcritical Bifurcation.** In order for

$$\dot{x} = f(x, \mu)$$

to undergo a transcritical bifurcation, we must have $f(0,0) = 0; D_x f(0,0) = 0$ (nonhyperbolic fixed point), $D_\mu f(0,0) = 0$ (existence of two curves), and $D^2_{x\mu}(0,0) \neq 0, D^2_{xx}(0,0) \neq 0$ (slope of the other curve).

**Remark:** The theorem show that the dynamics of our equation near $(0,0)$ are qualitiatively the same as

$$\dot{x} = \mu x \pm x^2$$

which can be viewed as the normal form for transcritical bifurcation.

## 1.4 The Pitchfork Bifurcation

In this case, (1) two curves of fixed points pass through $(0,0)$, one given by $x = 0$, the other by $\mu = x^2$; (2) the curve $x = 0$ exists on both sides of $\mu = 0$; the other curve exists on one side of $\mu = 0$; and (3) the fixed points on the curve $x = 0$ has different stability types on opposites of $\mu = 0$. The fixed points on $\mu = x^2$ all have the same stability type.

**Theorem on Pitchfork Bifurcation.** In order for

$$\dot{x} = f(x, \mu)$$

to undergo a pitchfork bifurcation, we must have $f(0,0) = 0; D_x f(0,0) = 0$ (nonhyperbolic fixed point), with $D_\mu f(0,0) = 0$ (existence of 2 curves), and $D^2_{xx} f(0,0) = 0$ (the other curve tangent to $\mu = 0$), $D^2_{x\mu}(0,0) \neq 0, D^3_{xxx}(0,0) \neq 0$ (the other curve on one side).

**Remark:** The theorem show that the dynamics of our equation near $(0,0)$ are qualitiatively the same as

$$\dot{x} = \mu x \pm x^3$$

which can be viewed as the normal form for pitchfork bifurcation.