# Discussion on KAM Theorem

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## 1 Integrable Systems

**Involution.** Consider Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

with integrals $F_1(q,p)$ and $F_2(q,p)$ (where $\frac{d}{dt}F_i = 0$). The functions $F_1$ and $F_2$ are in involution if

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial q}\frac{\partial F_2}{\partial p} - \frac{\partial F_1}{\partial p}\frac{\partial F_2}{\partial q} = 0.$$

Here, $\{,\}$ is the Poisson brackets.

Note: $F$ is an integral if $F$ and $H$ are in involution.

**Integrable Systems.** The $n$ degree of freedom Hamiltonian system is integrable if the system has $n$ integrals $F_1, \ldots, F_n$ which are functionally independent and in involution.

**Example:** The quadratic Hamiltonian generates an integrable system with integrals $F_i = (q_i^2 + p_i^2)$.

**Example:** If $H = H(p)$, the system is integrable with integrals $F_i = p_i$

**Symplectic Matrix and Canonical Transformation.** A matrix $M$ is symplectic if $M\Omega M^T = \Omega$. If the Jacobian of a transformation $((q,p) \to (Q,P))$ is symplectic, then it is a canonical transformation which preserves the Hamiltonian form of the equations.

**Action-Angle Variables.** These variables, which are related to polar coordinates, have the property that the transformation $p, q \to I, \theta$ is canonical. The transformation can be obtained by using the generating function $S(I,q)$, which has the property that

$$p = \frac{\partial S}{\partial q}, \qquad \theta = \frac{\partial S}{\partial I}.$$

**Theorem.** If a Hamiltonian system is integrable, then there exists a generating function $S(I,q)$ such that

$$H(p,q) = H(\frac{\partial S}{\partial q}, q) = H_0(I)$$

and the Hamiltonian system becomes

$$\dot{I} = 0, \qquad \dot{\theta} = \omega(I) = \frac{\partial H_0}{\partial I}.$$

**Resonance.** The frequencies $\omega_1, \ldots, \omega_n$ satisfies a resonance relation of order $k \in N$ if there exist $k_1, \ldots, k_n$ such that

$$k_1\omega_1 + \ldots + k_n\omega_n = 0$$

with $k = |k_1| + \ldots + |k_n|$.

## 2 Perturbation of Integrable Systems

**Nearly Integrable Systems.** Suppose that $H(p, q)$ has a small parameter $\epsilon$ and introduction of action-angle coordinates produces the system

$$\begin{aligned}\dot{I} &= \epsilon f(I, \theta) \\ \dot{\theta} &= \omega(I) + \epsilon g(I, \theta).\end{aligned}$$

If $\epsilon = 0$ the system is integrable; for small positive $\epsilon$ the system is called nearly integrable and can be studied by Birkhoff-Gustavson normalization. Suppose the new Hamiltonian is

$$H_0(I) + \epsilon H_1(I, \theta).$$

**KAM Theorem.** If $H_0$ is non-degenerate, i.e.

$$\det\ (\frac{\partial^2 H_0}{\partial I^2}) \neq 0,$$

then most of the invariant tori which exist for the unperturbed system will exist, abeit slightly deformed, for sufficiently small postive $\epsilon$; moreover, the measure of the complement of the set of tori tends to zero as $\epsilon$ tends to zero.