# CDS 205 Final Project: Dielectric Interactions with Electromagnetic Radiation

by Amir Safavi-Naeini, prepared for Prof J. E. Marsden

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## Contents

# 1 Introduction

Since the seminal work of Art Ashkin[1] in 1970 on the trapping and manipulation of dielectric particles by radiation forces, radiation pressure forces have been at the forefront of current experimental research. Optical tweezing, laser cooling of atoms, and the new field of optomechanically coupled devices are example incarnations of this field of study. A coherent and unified approach to dealing with these systems is however lacking.

The forces on the particle are usually split into two types, the "scattering force" and the "gradient force"[2]. In Ashkin's paper, and many studies of optical tweezing, the case of the gradient force on a dielectric sphere is treated through the use of ray optics and invocation of the conservation of momentum. For the case of the "scattering force", arguments are based on absorption and emission rates, and the directionality of each process along with the conservation of momentum .

On the other hand, studies of laser cooling of two- or multi-level systems use either the highly fruitful quantum mechanical formulation[1] of Claude Cohen-Tannoudji[4], or a perturbation theoretical approach.

More recent work, tracing originally back to the 1970s[3], and having been recently spurred by advances in nano-fabrication of optical cavities, usually start with the phenomonological hamiltonian

$$H = \omega_0(1 - gx)a^*a + \frac{\Omega^2}{2}x^2 + \frac{1}{2}p^2 \tag{1}$$

Where $a$ is the mode amplitude of the electromagnetic field. The use of this methodology has lead to remarkably close the precise agreement with measurement. It is not difficult to see that this modal Hamiltonian can only be an approximation of the full equations of motion.

[1]The *Atome Habillé* or "dressed atom" approach for which the Nobel Prize in Physics of 1997 was awarded to Claude Cohen-Tannoudji.

It is of interest to develop the full equations for dielectric-radiation interactions so that all of the above treatments can be placed in an equivalent, general footing, and that limitations in the usual hamiltonians, such as the one above can be identified.

The first step in such a program is the development of variational principles for the "macroscopic" maxwell equations. This is done in section 2, and constitutes the heart of this article. It is therein argued that it may be proper to view $A$ and $\phi$ as positions, the "electric field" $E$ and "magnetic field" $B$ as velocities, and the "electric displacement" $D$ and "magnetizing field" $H$ as momenta. With respect to the correct field Lagrangian, the four Maxwell's equations can then be interpreted as two kinematic relations (akin to $\dot{q} = v$) and two Euler-Lagrange equations. The two "constitutive relations", are then only the definition of canonical momenta. This treatment is made particularly lucid through the use of differential forms which make $D(B)$ and $E(H)$ different geometrical objects, 2- and 1-forms respectively.

# 2 Variational Formulation of Microscopic and Macroscopic Maxwell's Equations

## 2.1 Vector Calculus Maxwell's Equations in 3+1 Dimensions

Electromagnetic theory is usually used by physicists, engineers and experimentalists who may not be familiar with the differential forms. There are many resources for learning about differential forms[6] and in particular, its application to electromagnetics[5]. We start at first with the vector calculus formulation of Maxwell's equations which is what is usually found in the relevant physics texts and literature:

$$
\begin{aligned}
& \text{Maxwell-Faraday} \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \qquad (2) \\
\nabla \cdot \mathbf{B} &= 0 \qquad (3) \\
& \text{Maxwell-Ampére} \\
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J} \qquad (4) \\
\nabla \cdot \mathbf{D} &= \rho \qquad (5) \\
& \text{General Constitutive Relations} \\
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P}[\mathbf{E}, \mathbf{B}, t] \qquad (6) \\
\mathbf{H} &= \mu_0^{-1} \mathbf{B} - \mathbf{M}[\mathbf{E}, \mathbf{B}, t] \qquad (7)
\end{aligned}
$$

The vectors $\mathbf{P}[\mathbf{E}, \mathbf{B}, t]$ and $\mathbf{M}[\mathbf{E}, \mathbf{B}, t]$ are called the polarization and magnetization vectors respectively and are used to model the response of materials to the electromagnetic fields. They are are generally nonlocal in time and space.

The case were $\mathbf{P} = 0$ and $\mathbf{M} = 0$ is called the Microscopic Maxwell Equations.

An equivalent viewpoint involves the definition of "effective" or "bound" charges and currents which create the material response. Then the material response is modeled by simply adding $\mathbf{J}_b[\mathbf{E}, \mathbf{B}, t]$ and $\rho_b[\mathbf{E}, \mathbf{B}, t]$ to the microscopic Maxwell-Ampére equations:

$$
\begin{aligned}
\nabla \times \mu_0^{-1} \mathbf{B} - \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} &= \mathbf{J} + \mathbf{J}_b[\mathbf{E}, \mathbf{B}, t] \qquad (8) \\
\nabla \cdot \epsilon_0 \mathbf{E} &= \rho + \rho_b[\mathbf{E}, \mathbf{B}, t] \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\rho_b[E, B, t] &= -\nabla \cdot \mathbf{P}[\mathbf{E}, \mathbf{B}, t] \qquad (10) \\
\mathbf{J}_b[\mathbf{E}, \mathbf{B}, t] &= \nabla \times \mathbf{M}[\mathbf{E}, \mathbf{B}, t] - \frac{\partial}{\partial t} \mathbf{P}[\mathbf{E}, \mathbf{B}, t] \qquad (11)
\end{aligned}
$$

#### 2.1.1 A Variational Principle for the Microscopic Maxwell's Equations

The Lagrangian is a function $L : TQ \to R$. In this section, we'll use a slightly looser definition of a Lagrangian which will simplify much what will be done later.

Taking the cue from the usual lagrangian $L(q, \dot{q})$ which is a function the position $q$, and a variable kinematically related to $q$, i.e. $\dot{q}$ which may be found by a differentiation, we generalize this differentiation to spatial variables as well and define our Lagrangian density to be $\mathcal{L}(\mathbf{A}, \phi, -\dot{\mathbf{A}} - \nabla\phi, \nabla \times \mathbf{A}) = \mathcal{L}(\mathbf{A}, \phi, \mathbf{E}, \mathbf{B})$. The action is then given by:

$$S[\mathbf{A}, \phi, \mathbf{E}, \mathbf{B}] = \int\int \mathcal{L}(\mathbf{A}, \phi, \mathbf{E}, \mathbf{B}) d^3\mathbf{x}dt \tag{12}$$

$$\mathcal{L}(\mathbf{A}, \phi, \mathbf{E}, \mathbf{B}) = \frac{\epsilon_0 \mathbf{E} \cdot \mathbf{E} - \mu_0^{-1} \mathbf{B} \cdot \mathbf{B}}{2} - \rho\phi + \mathbf{J} \cdot \mathbf{A} \tag{13}$$

This formulation is much more transparent the 4-vector notation. In any case Euler-Lagrange equations for the above lagrangian density can be shown to be:

$$\nabla \times \mu_0^{-1} \mathbf{B} - \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} = \mathbf{J} \tag{14}$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho \tag{15}$$

This is proven in section A.2 using the language of differential forms, which is introduced in the next section. These are none other than the Maxwell-Ampére equations for vacuum. The Maxwell-Faraday equations are implicit in our definition of $\mathbf{E}$ and $\mathbf{B}$.

## 2.2 Differential Forms Maxwell's Equations in 3+1 Dimensions

Before going any further, it is fruitful to begin using the differential forms formulation of Maxwell's equations. Not only only will this will make the derivations more mathematically elegant, it will also make them more physically meaningful. In the vector calculus formulation of Maxwell's equations, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$ and $\mathbf{M}$ are all vector fields and hence mathematically indistinguishable. Yet they are physically different quantities. By defining them to be different types of differential forms, one can not only avoid mistakes, but elucidate the derivations and provide motivations for definitions.

For the Maxwell-Faraday equations, we define a 0-form $\phi$ and a 1-form $A$. Then the 1-form $E$ and 2-form $B$ are:

$$E = -\partial_t A - \mathbf{d}\phi \tag{16}$$

$$B = \mathbf{d}A \tag{17}$$

From here arrive at

$$\begin{aligned}\mathbf{d}E &= -\partial_t \mathbf{d}A - \mathbf{dd}\phi &(18)\\ &= -\partial_t B &(19)\\ \mathbf{d}B &= \mathbf{dd}A = 0 &(20)\end{aligned}$$

These are the first two of Maxwell's equations, the Maxwell-Faraday equations.

#### 2.2.1 Motivation for the Definition of $D$ and $H$ for the Microscopic Case

Next we motivate what $D$ and $H$ must be by looking at the Lagrangian density. Since $\mathcal{L}$ from equation (13) gives us the Lagrangian $L$ under an integral $\int d^3\mathbf{x}$, we know that it must be a 3-form.

This means:

$$\mathcal{L}(A, \phi, E, B) = \frac{\epsilon_0 * E \wedge E - \mu_0^{-1} * B \wedge B}{2} - \rho \wedge \phi + J \wedge A \qquad (21)$$

Note that this naturally defines $\rho$ to be a 3-form (as expected, since a volume integral gives us the charge) and $J$ to be a 2-form (since a surface integral gives us the current).

We define $D$ and $H$ as "conjugate momenta"[2]:

$$\begin{aligned}D &= \frac{\partial \mathcal{L}}{\partial E} = \epsilon_0 * E &(22)\\ H &= -\frac{\partial \mathcal{L}}{\partial B} = \mu_0^{-1} * B &(23)\end{aligned}$$

We see that $D$ and $H$ can be thought of as playing the role of "conjugate momenta" to $A$ and $\phi$ while $E$ and $B$ play the role of "velocities"[3]. We also see that the constitutive relations are none other than an analog to $\partial L/\partial v = p$. This is particularly pleasing because the $E$ and $B$ are kinematically related to $A$ and $\phi$, while $D$ and $H$ provide us with the dynamical relations of the

[2]Deschamps[5] also defines $D = \epsilon * E$ and $H = \mu^{-1} * B$ though he does not use a variational principle.

[3]This is different than what is usually considered to be the conjugate momentum to $\mathbf{A}$, i.e. $\mathbf{Y} = -\mathbf{E}$.

system. These dynamical relations are the Euler-Lagrange equations for the system (see section A.2 for derivations):

$$\frac{\partial \mathcal{L}}{\partial A} = -\mathbf{d}\frac{\partial \mathcal{L}}{\partial B} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial E} \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\mathbf{d}\frac{\partial \mathcal{L}}{\partial E} \tag{25}$$

Using $\partial\mathcal{L}/\partial\phi = -\rho$, $\partial\mathcal{L}/\partial A = J$ and equations (22), (23) above, these equations become:

$$\mathbf{d}H = J + \frac{d}{dt}D \tag{26}$$

$$\mathbf{d}D = \rho \tag{27}$$

These two equations are the Maxwell-Ampére equations.

### 2.2.2 Motivation for the Definition of $D$ and $H$ in the General Case

For the general case where the material response may be thought of currents and charges imposed from the outside, we may write the the 3-form for bound charges $\rho_b = -\mathbf{d}P$ and the 2-form for bound currents as $J_b = \mathbf{d}M - \frac{d}{dt}P$ following equations (10)-(11).

Adding these to the Lagrangian density (21), one finds:

$$\mathcal{L}(A,\phi,E,B) = \frac{\epsilon_0 * E \wedge E - \mu_0^{-1} * B \wedge B}{2} - \rho\wedge\phi - \rho_b\wedge\phi + J\wedge A + J_b\wedge A \tag{28}$$

Integration by parts changes this Lagrangian density to

$$\mathcal{L}(A,\phi,E,B) = \frac{\epsilon_0 * E \wedge E - \mu_0^{-1} * B \wedge B}{2} - \rho\wedge\phi + J\wedge A + P\wedge E + M\wedge B \tag{29}$$

Now using the previous definitions for $D$ and $H$, equations (22)-(23) become[4]:

$$D = \frac{\partial \mathcal{L}}{\partial E} = \epsilon_0 * E + P \tag{30}$$

$$H = -\frac{\partial \mathcal{L}}{\partial B} = \mu_0^{-1} * B - M \tag{31}$$

[4] This is in analogy to the situation arrived at when treating a charged particle in an external imposed field $\mathbf{A}$ with a velocity term $\mathbf{v}\cdot\mathbf{A}$ in the lagrangian where one finds $\mathbf{p} = m\mathbf{v} + \mathbf{A}$.

### 2.2.3 Summary of Analogy

The summarize the analogy, we compare the above definitions and derivations to the case of a single particle lagrangian $L(q, \dot{q})$.

The original Maxwell equations (2)-(7) came in three parts. The Maxwell-Faraday relations were shown to result from the kinematical relation between $E$, $B$ and $A$, $\phi$. This is analogous to $v = \dot{q}$.

The constitutive relations, were shown to be none other than the definition of conjugate momenta. In other words, they are the analogue to $p = \frac{\partial L}{\partial v}$.

Finally, the Maxwell-Ampére relations, are the dynamical relations, which were shown to be equivalent to the Euler-Lagrange equations of the Lagrangian Density.

$$\begin{aligned}
\text{Maxwell-Faraday} \quad &| \quad \text{Definition of Velocities} \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad & \quad E = -\partial_t A - \mathbf{d}\phi \qquad (32) \\
\nabla \cdot \mathbf{B} = 0 \quad & \quad B = \mathbf{d}A \qquad (33) \\
\text{General Constitutive Relations} \quad &| \quad \text{Canonical Momenta} \\
\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad & \quad D = \frac{\partial \mathcal{L}}{\partial E} \qquad (34) \\
\mathbf{H} = \mu_0^{-1} \mathbf{B} - \mathbf{M} \quad & \quad H = -\frac{\partial \mathcal{L}}{\partial B} \qquad (35) \\
\text{Maxwell-Ampére} \quad &| \quad \text{Euler-Lagrange Equations} \\
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad & \quad \frac{\partial \mathcal{L}}{\partial A} = -\mathbf{d}\frac{\partial \mathcal{L}}{\partial B} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial E} \qquad (36) \\
\nabla \cdot \mathbf{D} = \rho \quad & \quad \frac{\partial \mathcal{L}}{\partial \phi} = -\mathbf{d}\frac{\partial \mathcal{L}}{\partial E} \qquad (37)
\end{aligned}$$

## 2.3 Hamilton-Pontryagin Description of the Electromagnetic Field in Vacuum

The Hamilton-Pontryagin description gives one a functional which when extremized provides the definition of conjugate momenta, Hamilton's equations, the Euler-Lagrange equations, and their relation through the legendre transform. This is done by introducing an auxiliary variable $\mathbf{v}$ and optimization of the functional

$$S_{\text{HP}}[\mathbf{q},\mathbf{v},\mathbf{p}] = \int L(\mathbf{q},\mathbf{v}) + \mathbf{p}\cdot(\dot{\mathbf{q}}-\mathbf{v})dt \tag{38}$$

One is in effect "hard-coding" the relations $\dot{\mathbf{q}} = \mathbf{v}$, $\frac{\partial L}{\partial \dot{\mathbf{q}}} = \mathbf{p}$ into the problem. In the same way, one can "hard-code" the kinematical relations $E = -\partial_t A - \mathbf{d}\phi$ and $B = \mathbf{d}A$.

Following the discussion in section 2.2.3, one finds in complete analogy that the correct Hamilton-Pontryagin action is given by:

$$S_{\text{HP}}[A,\phi,E,B,D,H] = \int\int \mathcal{L}_{\text{HP}} dt \tag{39}$$

$$\begin{aligned} \mathcal{L}_{\text{HP}}(A,\phi,E,B,D,H) \quad = \quad & \mathcal{L}(A,\phi,E,B) \\ & +D\wedge(-\partial_t A - \mathbf{d}\phi - E) \\ & -H\wedge(\mathbf{d}A - B) \end{aligned} \tag{40}$$

The negative sign on $H$ is due to its definition as $H = -\frac{\partial \mathcal{L}}{\partial B}$. A similar Hamilton-Pontryagin principle for the microscopic Maxwell equations can be found in Schwinger's Electrodynamics text[7][5].

Extremizing with respect to variations in $A$, $\phi$, $E$, $B$, $D$ and $H$ of $S_{\text{HP}}[A,\phi,E,B,D,H]$ give the Maxwell-Ampére, the constitutive and the Maxwell-Farday equations respectively.

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[5]Schwinger does not extend this to the macroscopic Maxwell equations. This mainly because Schwinger was a member of the electromagnetics community in the 1940-50s at the MIT Radiation Lab where most applications involved either waveguides or radar scattering where the dielectric response of materials was not particularly important or interesting. Also, his Nobel prize winning work was in quantum electrodynamics, where one only considers the microscopic Maxwell equations. Only with the gradual unification of optics and electromagnetics has the dielectric response of materials gained practical prominence.

Schwinger also does not identify explicitly $E$ and $B$ as velocities, possibly because of the confusion which arises from also identifying $E$ as a momentum. The different roles of $E$ and $D$, which has to do with them being different types of geometrical objects only becomes clear in the differential forms language.

Interestingly, he calls the general $S_{\text{HP}}[\mathbf{q},\mathbf{v},\mathbf{p}]$ shown in equation (38), the "third way" variational principle, as opposed to "first" and "second" ways which are the hamiltonian and lagrangian formulations.

# 3 Radiation-Matter Interaction

We are now ready to apply the variational principle developed in the previous section to the problem at hand, the interaction of a dynamic dielectric with radiation. To find the total Lagrangian of the system, we add the lagrangian of the material system to that of the field, and enforce the interaction through the term $P \wedge E$ in the Lagrangian density.

Then the total Lagrangian can be broken up into three terms:

$$\begin{aligned}
& & \text{Field Lagrangian} \\
L^{\mathrm{f}} &=& \int \mathcal{L}^{\mathrm{f}}(A, \phi, E, B) \qquad (41) \\
& & \text{Interaction Lagrangian} \\
L^{\mathrm{int}} &=& \int \mathcal{L}^{\mathrm{int}} = \int P \wedge E \qquad (42) \\
& & \text{Material Lagrangian} \\
L^{\mathrm{mat}} &=& L^{\mathrm{mat}}(\mathbf{q}^i, \mathbf{v}^i) \qquad (43) \\
& & \text{Interaction Condition} \\
P &=& P[\mathbf{q}^i, \mathbf{v}^i; A, \phi, E, B] \qquad (44)
\end{aligned}$$

There is some arbitrariness to this division. The general requirement is that the material variables don't make an appearance in the field lagrangian, and that the field variables avoid the material lagrangian.

## 3.1 Infinitesimally Small Dielectric Particle with Free Fields

The simplest system to treat is that of small, free dielectric particle. In this case, $L^{\mathrm{mat}} = \frac{1}{2}m\dot{\mathbf{q}}^2$.

We take this to be a "small", neutral, point particle. The particle of interest has a constant, dispersion-less dielectric susceptibility of $\chi_p$. In other words, $P(\mathbf{x}) = \chi_p \Theta(\mathbf{x} - \mathbf{q}) * E(\mathbf{x})$.

Using equation (42) and an assumption that the volume of the particle, $V_p$ is small enough,

$$L^{\mathrm{int}} = \chi_p V_p ||E||^2(\mathbf{q}) \qquad (45)$$

The equation of motion for the particle then becomes:

$$m\ddot{\mathbf{q}} = \chi_p V_p \nabla ||E||^2(\mathbf{q}) \tag{46}$$

This is the "gradient force" derived originally by Ashkin through use of geometrical optics.

## 3.2 Larger Dielectric Particles

The analysis of larger particles is difficult because the effects of interest arise from "self-interactions", in much the same way that Bremsstrahlung can be interpreted to occur due to interaction of a particle with the field created by itself. For the case of the infinitesimally small particle, one ignores the effect of the the particle on the fields, making the system very simple to analyse. The same cannot be said for the larger (order of wavelength) particles.

For larger dielectric particles, the interaction with $L^{\mathrm{f}}$ must also be taken into account. These interactions give rise to resonances, and hence memory. One possible path is to incorporate this into the dispersion of susceptibility. In these cases, though the gradient force remains essentially the same, there will also be "memory" in the total polarizability of the particle (caused by either Mie or atomic resonances). This memory in turn gives rise to a velocity dependent force. This velocity dependent force in turn gives rise to effects which make damping or laser cooling possible.

## 3.3 Deformable Cavities

Deformable optical cavities cause two seemingly difficult to handle complications. Firstly, because they are cavities, there will be a large memory in the effective polarizability of the system, arising from interactions with $L^{\mathrm{f}}$. These systems are deep in the "strongly interacting" regime. A small change in the cavity shape can have grave consequences for the future evolution of the fields.

Secondly, the material system itself is no longer described by the few degrees of freedom of a single particle, but by a displacement field $\mathbf{u}(\mathbf{x}, t)$ with its own dynamics.

# 4 Conclusions

In this article we studied a possible general framework for the analysis of the interaction of dielectric materials with the electromagnetic field. We start by extending slightly, the usual electromagnetic field lagrangian to take into account the material properties. This approach is made particularly transparent through the of differential forms and may be interesting in its own right because it illuminates some of the geometry which is present in Maxwell's equations.

From there, it is then very easy to incorporate the dynamics of the material itself. In section 3.1, we see how this method gives us the usual "gradient force".

For future it study in may be interesting to try applying this formalism to a strongly interacting system (i.e. particles with Mie resonances, or dielectric cavities). One path may be through the use of a Green's function formalism, where the Green's function is itself a function of the material configuration co-ordinates.

# A Derivations

## A.1 The partial derivative of the Lagrangian Density $\mathcal{L}(A, \phi, E, B)$

The 3-form $\mathcal{L}$ of is a function of the local $A$, $\phi$, $E$, and $B$ at a particular point on the manifold. We define the various partial derivatives as:

$$\frac{\partial \mathcal{L}}{\partial A} \wedge \alpha = \frac{d}{d\epsilon} \mathcal{L}(A + \epsilon\alpha, \phi, E, B)\bigg|_{\epsilon=0} \tag{47}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} \wedge \gamma = \frac{d}{d\epsilon} \mathcal{L}(A, \phi + \epsilon\gamma, E, B)\bigg|_{\epsilon=0} \tag{48}$$

$$\frac{\partial \mathcal{L}}{\partial E} \wedge \eta = \frac{d}{d\epsilon} \mathcal{L}(A, \phi, E + \epsilon\eta, B)\bigg|_{\epsilon=0} \tag{49}$$

$$\frac{\partial \mathcal{L}}{\partial B} \wedge \beta = \frac{d}{d\epsilon} \mathcal{L}(A, \phi, E, B + \epsilon\beta)\bigg|_{\epsilon=0} \tag{50}$$

$$\tag{51}$$

Where $\alpha$, $\gamma$, $\eta$ and $\beta$ are 1-, 0-, 1- and 2-forms respectively.

## A.2 Euler-Lagrange Equations for $\mathcal{L}(A, \phi, E, B)$

The E-L equations are derived for the action $S[A, \phi, E, B]$ by minimizing with respect to variations in $A$ and $\phi$.

### A.2.1 Variations $\delta\phi$ and the Divergence Equation

A variation in $\phi \to \phi + \delta\phi$ causes a corresponding variation in $E \to E - \mathbf{d}\delta\phi$. We are left with:

$$\delta S = \int\int \frac{\partial \mathcal{L}}{\partial E} \wedge (-\mathbf{d}\delta\phi) + \frac{\partial \mathcal{L}}{\partial \phi} \wedge \delta\phi dt \tag{52}$$

Using

$$\mathbf{d}\left(\frac{\partial \mathcal{L}}{\partial E} \wedge (\delta\phi)\right) = \left(\mathbf{d}\frac{\partial \mathcal{L}}{\partial E}\right) \wedge (\delta\phi) + \frac{\partial \mathcal{L}}{\partial E} \wedge (\mathbf{d}\delta\phi) \tag{53}$$

And assuming the integration area is large enough to make the field terms zero in the boundary,

$$\delta S = \int\int \left(\mathbf{d}\frac{\partial \mathcal{L}}{\partial E}\right) \wedge (\delta\phi) + \frac{\partial \mathcal{L}}{\partial \phi} \wedge \delta\phi dt \tag{54}$$

Setting the variation to 0, one arrives at:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\mathbf{d}\frac{\partial \mathcal{L}}{\partial E} \tag{55}$$

### A.2.2 Variations $\delta A$ and the Curl Equation

A variation in $A \to A + \delta A$ causes a corresponding variation in $E \to E - \partial_t \delta A$ and $B \to B + \mathbf{d}\delta A$. We are left with:

$$\delta S = \int\int \frac{\partial \mathcal{L}}{\partial E} \wedge (-\partial_t \delta A) + \frac{\partial \mathcal{L}}{\partial B} \wedge \mathbf{d}\delta A + \frac{\partial \mathcal{L}}{\partial A} \wedge (\delta A) dt \tag{56}$$

Using

$$\mathbf{d}\left(\frac{\partial \mathcal{L}}{\partial B} \wedge (\delta A)\right) = \left(\mathbf{d}\frac{\partial \mathcal{L}}{\partial B}\right) \wedge (\delta A) - \frac{\partial \mathcal{L}}{\partial B} \wedge (\mathbf{d}\delta A) \tag{57}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial E}\wedge(\delta A)\right)=\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial E}\right)\wedge(\delta A)+\frac{\partial \mathcal{L}}{\partial E}\wedge(\frac{d}{dt}\delta A) \tag{58}$$

Again, setting the boundary terms to be zero, one arrives at:

$$\delta S=\int\int\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial E}\right)\wedge(\delta A)+\left(\mathbf{d}\frac{\partial \mathcal{L}}{\partial B}\right)\wedge(\delta A)+\frac{\partial \mathcal{L}}{\partial A}\wedge(\delta A)dt \tag{59}$$

Setting the variation to zero, one arrives at:

$$-\mathbf{d}\frac{\partial \mathcal{L}}{\partial B} \quad = \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial E}+\frac{\partial \mathcal{L}}{\partial A} \tag{60}$$

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