ON WHITHAM'S AVERAGING METHOD
CDS 205 COURSE PROJECT

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1. Introduction to Whitham's Averaging Method

Whitham's averaging method is developed as a general technique for dispersive nonlinear waves [13, 8]. This kind of methods has significant connections with mathematics, physics and engineering [6], for example, algebraic geometry, plasma physics, fiber optics, and string theory.

The general idea is, when we consider a nonlinear periodic wave, we can decompose it into a 'slow' modulated wave and 'fast' oscillations. We can average the evolution equation over fast variables, and obtain equations for slow modulated parameters of the periodic (single phase) or quasi-periodic (multiphase) solutions of the PDE. This approach is due to Whitham [13] and the averaged modulational equations are called the Whitham equations.

The idea can be applied not only to integrable equations like KdV or NLS, but to any evolution equation having periodic solutions.

Whitham's method can be obtained by a multiple scale expansion type argument, however, it turns out that variational approach can give a more compact and significant derivation of the modulational equation. It's of interest to investigate the geometric structure of the Whitham's equation. For example, it has been proved in [5, 4] that the multiphase Whitham's equation for KdV equation ($\varphi_t = 6\varphi\varphi_x - \varphi_{xxx}$), and Sine-Gordon equation ($\varphi_{tt} - \varphi_{xx} = \sin\varphi$) has so-called 'Riemann invariants', which means that the Whitham's equations can be diagonalized and is therefore an essential reduction of the original system (a dimension≥ 3 system does not necessarily have Riemann invariants).

In the following section 2, we will present the geometric formulation of Whitham's method by B.A.Dubrovin and S.P.Novikov following [2, 3, 11] , in section 3, we will prove a rigorous justification of the leading order Whitham approximation for linear Klein-Gordon equation, which can also applied to nonlinear Klein-Gordon equation.

2. Geometric Mechanical Formulation of Whitham's Equation

2.1. **Lagrangian Formulation.** Suppose the evolution system has Lagrangian $L(q, q_x, q_t)$, (for example, Klein-Gordan, KdV, NLS, etc), generally, we can consider equations of hydrodynamic type ($q^i_t = v^{i\alpha}_j(u)q^j_\alpha, q^j_\alpha \equiv \frac{\partial q^j}{\partial x^\alpha}$), which includes larger class of equations, e.g., Euler equations for fluid, Yang-Mills etc.

Now suppose we have a m-phase periodic solution of the problem, in other words, let $q = Q(kx - \omega t, u^1, \ldots, u^{2m})$ be a function on the m-torus depending on $2m$ parameters $u^1, \ldots, u^{2m}$.

Define the averaged Lagrangian by averaging the Lagrangian $L(q, q_x, q_t)$ over each phase τ^i. Since $q_t = \omega Q_\tau$, $q_x = kQ_\tau$, we have,

$$\mathcal{L}(\tau, u) = (2\pi)^{-m} \int L(Q(\tau, u), kQ_\tau(\tau, u), \omega Q_\tau(\tau, u)) d^m\tau$$

Therefore, the equations of slow modulation waves can be obtained by the Euler-Lagrangian equation for $\int \mathcal{L}(k, \omega, u) dX dT$.

Variation of τ gives

$$\frac{\partial}{\partial X}\frac{\partial \mathcal{L}}{\partial k} - \frac{\partial}{\partial T}\frac{\partial \mathcal{L}}{\partial \omega} = 0 \tag{2.1}$$

Variation of u gives

$$\frac{\partial \mathcal{L}}{\partial u} = 0 \tag{2.2}$$

together with the compatible condition

$$k_T + \omega_X = 0 \tag{2.3}$$

Notice here k has the sense of wave number and ω has the sense of frequency, equation (2.2) has the meaning of dispersion relation. Also, formally, equation (2.3) is the conservation of wave number, and (2.1) is conservation laws of the modulated energy.

2.1.1. *Averaging Conservation Laws.* The Whitham's modulational equation can also be derived from averaging the conservation laws of the corresponding system. Suppose the system has N local field integrals

$$I_i = \int P_i(\varphi, \varphi_x, \varphi_{xx}, \dots) dx, i = 1, \dots, N$$

Let $Q_i(\varphi, \varphi_x, \varphi_{xx}, \dots)$ be the corresponding flux densities, i.e., for the solution of the original evolution equation, we have,

$$\frac{\partial P_i}{\partial t} = \frac{\partial Q_i}{\partial x}, i = 1, \dots, N$$

For example, the KdV equation has the following conservation laws,

$$\begin{aligned} u_t + (3u^2 + u_{xx})_x &= 0 \\ (\frac{1}{2}u^2)_t + (2u^3 + uu_x - \frac{1}{2}u_x^2)_x &= 0 \\ (u^3 - \frac{1}{2}u_x^2)_t + (\frac{9}{2}u^4 + 3u^2 u_{xx} + \frac{1}{2}u_{xx}^2 + u_x u_t) &= 0 \end{aligned}$$

Actually, it can be proved there exists hierarchy of such conservation laws for KdV equation.

Now average the conservation laws,

$$\begin{aligned} \bar{P}_i &= (2\pi)^{-m} \int P_i(\Phi(\tau, u), \dots) d^m\tau = I_i \\ \bar{Q}_i &= (2\pi)^{-m} \int Q_i(\Phi(\tau, u), \dots) d^m\tau \end{aligned}$$

therefore, the averaged conservation law is

$$\frac{\partial \bar{P}_i}{\partial T} = \frac{\partial \bar{Q}_i}{\partial X}$$

Remark 2.1. it can be proved that the form of the averaged equation using Lagrangian formulation is independent of the choice of conservation laws.

2.2. **Hamiltonian Structure.** Basically, the Hamiltonian structure can be obtained by Legendre transform from Lagrangian formulation. Introduce the function $S(X,T)$, such that $k(u(X,T)) = S_X$, $\omega(u(X,T)) = S_T$, which in the single phase case, is just the phase function. Now let the parameters u also depend on k, ω, the Lagrangian is then $\hat{\mathcal{L}} = \hat{\mathcal{L}}(S_X, S_T) = \mathcal{L}(k, \omega, u(k, \omega))$.

Perform the Legendre transform

$$(S, S_T) \to (S, J = \frac{\partial \hat{\mathcal{L}}}{\partial S_T})$$

$$\mathcal{H} = \mathcal{H}(S_X, J) = JS_T - \hat{\mathcal{L}}(S_X, S_T), S_T = S_T(J, S_x)$$

Write in variables (J,$k = S_x$), the slow modulation equations are in Hamiltonian form with $\mathcal{H}(k, J)$

$$\begin{aligned} k_T &= \partial_X \mathcal{H}_J \\ J_T &= \partial_X \mathcal{H}_k \end{aligned}$$

Suppose the solution is single phase, we can compute J like

$$\begin{aligned} J &= (2\pi)^{-1} \oint L(Q, kQ_\tau, \omega Q_\tau) d\tau = (2\pi)^{-1} \oint Q_\tau \frac{\partial L}{\partial q_t} d\tau \\ &= (2\pi)^{-1} \oint p dq \end{aligned}$$

thus, J is the action variable canonically conjugate to the angle variable τ, which is also the classical result of adiabatic invariants.

Note that the existence of multiphase solution of the original evolution equation can be obtained from this action-angle argument.

2.3. **Averaged Poisson Brackets.** It's of interest that we can actually directly average the Poisson brackets of the corresponding system, and the averaged system is Hamiltonian with respect to the averaged brackets, which is equivalent to the Whitham's system.

The Poisson bracket is defined in [2]

Definition 2.2. Poisson brackets of hydrodynamic type are local Poisson brackets of the form

$$\{u^q(x), u^p(y)\} = g^{qp}(u(x))\delta'(x - y) + b_s^{qp}(u(x))u_x^s \delta(x - y) \tag{2.4}$$

Here g^{qp} and b_s^{qp} are smooth functions in local coordinates on the u-space, which is a finite-dimensional manifold M. For the moment, the expression (2.4) is to be interpreted formally. With brackets of hydrodynamic type, Hamiltonians of hydrodynamic type ($H = \int h(u)dx$) generate equations of hydrodynamic type of the form $u_t^i = \{u^i(x), H\}$.

More precisely, the Poisson bracket of any two functionals $I_1(u)$, $I_2(u)$ has the form

$$\{I_1, I_2\} = \int dx (\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)})$$

where

$$A = (A^{qp}) = (g^{qp}(u)\frac{d}{dx} + b_s^{qp}(u)u_x^s)$$

For example, the KdV bracket write

$$\{I_1, I_2\} = \int_{-\infty}^{\infty} \frac{\delta I_1}{\delta u} \frac{d}{dx} (\frac{\delta I_2}{\delta u})$$

and d/dx represents the total derivative with respect to x, therefore, for KdV, $\{u(x), u(y)\} = \delta'(x-y)$

Now we have Novikov's results:
suppose we have N commuting local integrals I_i (for example, in exercise 3.3.1 [10]), $\{I_i, I_j\} = 0$, $I_i = \int P_i dx$, then we have $\{P_i(x), P_j(y)\} = \sum_k A_k^{ij} \delta^{(k)}(x-y)$, where $\delta^{(k)}(x-y)$ is the kth order distributive derivative of δ function. By the commuting property $\{I_i, I_j\} = 0$, $\int A_0^{ij} dx = 0$, therefore exists G^{ij}, such that $\partial_x G^{ij} = A_0^{ij}$.
Now introduce the averaged metric $g^{ij}(u)$ and connection $b_k^{ij}(u)$

$$\begin{aligned} g^{ij}(u) &= (2\pi)^{-m} \int A_1^{ij}(Q, Q', \dots) d^m \tau \\ b_k^{ij}(u) &= \frac{\partial}{\partial u^k} (2\pi)^{-m} \int G^{ij}(Q, Q', \dots) d^m \tau \end{aligned}$$

The averaged Poisson brackets can be defined as

$$\{u^i(X), u^j(Y)\} = g^{ij}(u(X))\delta'(X-Y) + b_k^{ij}(u(X)) u_x^k \delta(X-Y)$$

for $i, j = 1, \cdots, N$. The equations of slow modulation are Hamiltonian with respect to the averaged Poisson brackets
and averaged Hamiltonian $\bar{H} = \int [(2\pi)^{-m} \int h(Q, Q', \dots) d^m \tau] dX$

3. Mathematical Justification of Leading Order Term

There are several articles [9, 12, 7, 1] considering the justification of Whitham's modulation equation, the method we are using [9] is based on the the stability of the periodic solution and hyperbolicity of the modulation equation.
Let's consider the linear Klein-Gordon equations

$$u_{tt} - u_{xx} - u = 0$$

which is the E-L equation for the Lagrangian

$$\delta \int \int \frac{1}{2}(u_t^2 - u_x^2 - u^2) dx dt$$

introduce small parameter ε from, for example, initial condition, $U = a(X,T)\cos\theta$, where $X = \varepsilon x$, $T = \varepsilon t$, $\theta = \frac{\Theta(X,T,\varepsilon)}{\varepsilon}$, a and Θ are slowly varying.
The averaged Lagrangian is given by

$$\mathcal{L}(\omega, k, A) \quad = \quad \frac{1}{2\pi} A(\omega^2 - k^2 - 1) + O(\varepsilon^2), \ A = \frac{a^2}{2}$$

Notice A is the averaged Hamiltonian in general, thus, the variational equations are

$$\delta \int \int \mathcal{L}(\omega, k, A) dX dT = 0$$

And, let $\omega = -\theta_t$, $k = \theta_x$, the E-L equations are

$$\delta A: \qquad \mathcal{L}_A = 0$$
$$\delta\theta: \qquad \frac{\partial}{\partial T}\mathcal{L}_\omega - \frac{\partial}{\partial X}\mathcal{L}_k = 0$$

$\mathcal{L}_A = 0$ is $\omega^2 = k^2 + 1$, which is the dispersion relation for linear Klein-Gordon wave.

δ_θ gives $\frac{\partial}{\partial X}(kA) + \frac{\partial}{\partial T}(\omega A) = 0$, which is the conservation laws corresponding to wave energy.

together with the compatible condition $\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0$

We can reduce the modulation equation to $\frac{\partial k}{\partial T} + \omega'(k)\frac{\partial k}{\partial X} = 0$, and $\frac{\partial A}{\partial T} + \frac{\partial}{\partial X}(\omega'(k)A) = 0$, this is a conservation law system, however, it's degenerate in the sense that both wave and energy propagated with the group velocity $\omega'(k)$, given proper initial condition, we can solve it explicitly by characteristics, and prove that A has asymptotics $A \sim t^{-1/2}$

Suppose that $U = a\cos\theta$ and u satisfy the same initial condition, a,θ satisfy the modulation equation, and u satisfy the linear Klein-Gordon equation.

Suppose the residue $R = (U-u)/\varepsilon^{2-\gamma}$, $0 < \gamma < 2$

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial a}{\partial X}\varepsilon\cos\theta - ak\sin\theta \\
\frac{\partial^2 U}{\partial x^2} &= \varepsilon^2\frac{\partial^2 a}{\partial X^2}\cos\theta - \varepsilon\frac{\partial a}{\partial X}k\sin\theta - \varepsilon\frac{\partial(ak)}{\partial X}\sin\theta - ak^2\cos\theta \\
\frac{\partial U}{\partial t} &= \frac{\partial a}{\partial T}\varepsilon\cos\theta + a\omega\sin\theta \\
\frac{\partial^2 U}{\partial t^2} &= \varepsilon^2\frac{\partial^2 a}{\partial T^2}\cos\theta + \varepsilon\frac{\partial a}{\partial T}\omega\sin\theta + \varepsilon\frac{\partial(a\omega)}{\partial X}\sin\theta - a\omega^2\cos\theta
\end{aligned}$$

thus, let $L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - 1$ be the differential operator,

$$LR = \varepsilon^\gamma(\frac{\partial^2 a}{\partial X^2} - \frac{\partial^2 a}{\partial T^2}) = \varepsilon^\gamma N = \mathcal{O}(\varepsilon^\gamma)$$

where $|N|$ is bounded from above.

Now let $S = \partial_x R$, the linear part of the equation for R is $\partial_t(R,S) = G(R,S) = (S, \partial_x^2 R - R)$,

By Hille-Yosida theorem, the associate semigroup of G is a contraction semigroup with respect to the energy norm $\|(R,S)\|_Y^2 = \int((\partial_X R)^2 + R^2 + S^2)dx$, $Y = H^1(\mathbb{R}) \times L^2(\mathbb{R})$

Therefore, we have the exact form of residue $R(t) = G(t)R(0) + \varepsilon^\gamma\int_0^t G(t-s)N(s)ds$

suppose $|N(s)| \le C$, by Gronwall inequality, we have

$$\|R(t)\| \le C_1\exp(C\varepsilon^\gamma t)$$

If $\varepsilon^\gamma t \le T_0$, we have $\|U-u\|_Y \le C_1\exp(CT_0)\varepsilon^{2-\gamma}$,which means that the error of modulational solution is uniformly bounded by $\mathcal{O}(\varepsilon^{2-\gamma})$ up to time $\mathcal{O}(\varepsilon^{-\gamma})$

Thus, we have the theorem,

Theorem 3.1. *For linear Klein-Gordon equation, if the solution u and modulated solution U satisfies the same initial condition, then up to time $\varepsilon^{-\gamma}$, the error is* *uniformly bounded by*

$$\|u - U\|_Y \leq \varepsilon^{2-\gamma}$$

Notice that if we can prove the boundedness of the residue term, which is given by the stability of the modulational equation solution, this argument can be readily applied to the nonlinear case. In principle, it can also applied to justify the higher order approximation, in case we know the stability of corresponding system.

Remark 3.2. this method is from [9] with the same question, but with different scaling. Consider the solution of the form$U = a(X,T)e^{\imath(kx-\omega t)/\varepsilon}$, the slow time $T = \varepsilon^2 t$, and slow space variable $X = \varepsilon(x - \omega'(k)t)$, we can found that the amplitude function a satisfies NLS $(-2\imath\omega\partial_T a - \partial_X^2 a + \frac{1}{2}|a|^2 a = 0)$. By the fact KGE has a unique solution in Y, NLS has a unique solution in $H^1(\mathbb{R})$, and the corresponding residue term is of order $\mathcal{O}(\varepsilon^{\frac{7}{2}})$, the error estimate can be $\|u - U\|_Y = \mathcal{O}(\varepsilon^{\frac{3}{2}})$ up to $\mathcal{O}(\varepsilon^{-2})$

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