Simon- A good job on this problem. It is a good exercise to learn from in the general problem of going from a model with links to a chain or rope. We should continue to discuss that. Can you give me an extra copy of this to send to Scheurle? He is interested

The Triple Spherical Pendulum

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Abstract

In this report, the derivation and analysis of the triple-spherical pendulum is considered. A Lagrangian approach is taken and the relative equilibria of the system, with the condition that all the beads are pointing down, are found. The stability of these relative equilibria are then considered by applying the energy-momentum method, and results showed that there are two bifurcation parameters and four relative equilibrium solutions, of which one is the trivial rotation and the remaining are "cowboy" solutions. In particular, for the case of $r = 1$, $\hat{r} = 1$, $m = 2$, and $\hat{m} = 1$, there is only one stable "cowboy" solution and it resembles the double-spherical pendulum case in which the upper two beads portion acts like a single string with a mass.

1 Introduction

In the article by Marsden and Scheurle [1], the dynamical features of the double spherical pendulum was studied using Lagrangian reduction and bifurcation theory. The next simplest case in the pendulum family is the triple-beads case. In this report, the triple spherical pendulum case is considered (Figure 1).

2 Derivation

To analyze the dynamics of the triple spherical pendulum, a similar approach to Marsden and Scheurle [1] is taken. The configuration space is $Q = S^2{}_{l_1} \times S^2{}_{l_2} \times S^2{}_{l_3}$ with the constraints $\|\mathbf{q}_i\| = l_i$ where $\mathbf{q}_i \in S^2{}_{l_i}$ and l_i denotes the length of

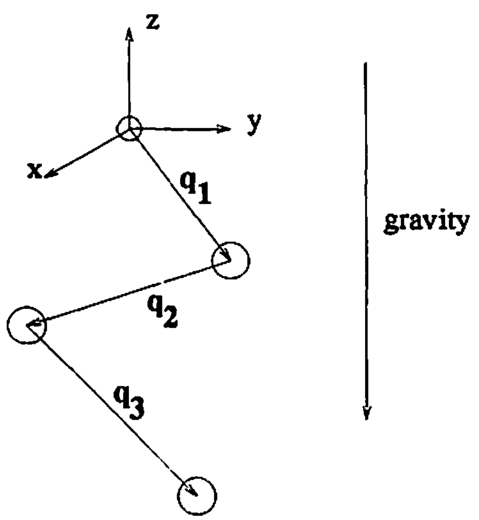

Figure 1: Triple spherical pendulum set up

the i^{th} string for $i = 1, 2, 3$. The derivation starts with the Lagrangian:

$$\begin{aligned} L(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2, \dot{\mathbf{q}}_3) \;=\; & \frac{1}{2}m_1\|\dot{\mathbf{q}}_1\|^2 + \frac{1}{2}m_2\|\dot{\mathbf{q}}_2\|^2 + \frac{1}{2}m_3\|\dot{\mathbf{q}}_3\|^2 - \\ & m_1 g \mathbf{q_1}\cdot\mathbf{k} - m_2 g(\mathbf{q_1}+\mathbf{q_2})\cdot\mathbf{k} - \\ & m_3 g(\mathbf{q_1}+\mathbf{q_2}+\mathbf{q_3})\cdot\mathbf{k} \end{aligned} \tag{1}$$

where $\mathbf{k}$ is the unit vector pointing in the z-direction. The conjugate momenta are defined to be:

$$\mathbf{p_i} = \frac{\partial L}{\partial \dot{\mathbf{q}}_i} \qquad \text{for} \quad i = 1, 2, 3 \tag{2}$$

As a result, the Hamiltonian under the Legendre transformation is:

$$\begin{aligned} H &= \sum_{i=1}^{3} \mathbf{p_i}\mathbf{q_i} - L(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2, \dot{\mathbf{q}}_3) \\ &= \frac{1}{2m_1}\|\mathbf{p_1}-\mathbf{p_2}\|^2 + \frac{1}{2m_2}\|\mathbf{p_2}-\mathbf{p_3}\|^2 + \frac{1}{2m_3}\|\mathbf{p_3}\|^2 + \\ & \quad m_1 g\mathbf{q_1}\cdot\mathbf{k} + m_2 g(\mathbf{q_1}+\mathbf{q_2})\cdot\mathbf{k} + m_3 g(\mathbf{q_1}+\mathbf{q_2}+\mathbf{q_3})\cdot\mathbf{k} \end{aligned} \tag{3}$$

The action of simultaneous rotation of all three spheres through an angle θ is symmetric about the z-axis. In other words, if R_θ is defined to be the action of rotation through an angle θ, then the action

$$(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) \mapsto (R_\theta\mathbf{q_1}, R_\theta\mathbf{q_2}, R_\theta\mathbf{q_3})$$

is symmetric. Let the rotation vector be $\omega\mathbf{k}$ where $\omega \in \mathbb{R}$. The infinitesimal generator corresponding to this rotation vector is:

$$\omega(\mathbf{k}\times\mathbf{q_1}, \mathbf{k}\times\mathbf{q_2}, \mathbf{k}\times\mathbf{q_3})$$

The corresponding conserved quantity is the angular momentum about the z-axis:

$$\begin{aligned} < J(\mathbf{q_1},\mathbf{q_2},\mathbf{q_3},\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}), \omega\mathbf{k} > &= \omega\,[\mathbf{p_1}\cdot(\mathbf{k}\times\mathbf{q_1}) + \mathbf{p_2}\cdot(\mathbf{k}\times\mathbf{q_2}) + \mathbf{p_3}\cdot(\mathbf{k}\times\mathbf{q_3})] \\ &= \omega\mathbf{k}\cdot[\mathbf{q_1}\times\mathbf{p_1} + \mathbf{q_2}\times\mathbf{p_2} + \mathbf{q_3}\times\mathbf{p_3}] \\ \Rightarrow J &= \mathbf{k}\cdot[\mathbf{q_1}\times\mathbf{p_1} + \mathbf{q_2}\times\mathbf{p_2} + \mathbf{q_3}\times\mathbf{p_3}] \end{aligned} \quad (4)$$

Substituting equation (2) for $i = 1, 2, 3$ and rearranging:

$$\begin{aligned} J &= \mathbf{k}\cdot[m_1(\mathbf{q_1}\times\dot{\mathbf{q_1}}) + m_2(\mathbf{q_1}+\mathbf{q_2})\times(\dot{\mathbf{q_1}}+\dot{\mathbf{q_2}}) + \\ &\quad m_3(\mathbf{q_1}+\mathbf{q_2}+\mathbf{q_3})\times(\dot{\mathbf{q_1}}+\dot{\mathbf{q_2}}+\dot{\mathbf{q_3}})] \end{aligned} \quad (5)$$

For simple systems, the locked inertia tensor ($\mathbb{I}$) is the moment of inertia of the system treated as a rigid system. For the case of the triple spherical pendulum, the locked inertia tensor is calculated to be:

$$\mathbb{I}(\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}) = m_1\|\mathbf{q_1^\perp}\|^2 + m_2\|\mathbf{q_1^\perp}+\mathbf{q_2^\perp}\|^2 + m_3\|\mathbf{q_1^\perp}+\mathbf{q_2^\perp}+\mathbf{q_3^\perp}\|^2$$

where $\|\mathbf{q_i^\perp}\|^2 = \|\mathbf{q_i}\|^2 - \|\mathbf{q_i}\cdot\mathbf{k}\|^2$ for $i = 1, 2, 3$, i.e. the projected length of the vector $\mathbf{q_i}$ on the x-y plane. With this expression for the locked inertia tensor, the amended potential (V_μ) can be obtained as follows:

$$\begin{aligned} V_\mu(\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}) &= V(\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}) + \frac{1}{2} < \mu, \mathbb{I}^{-1}(\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}) > \\ &= m_1 g\mathbf{q_1}\cdot\mathbf{k} + m_2 g(\mathbf{q_1}+\mathbf{q_2})\cdot\mathbf{k} + m_3 g(\mathbf{q_1}+\mathbf{q_2}+\mathbf{q_3})\cdot\mathbf{k} + \\ &\quad \frac{1}{2}\frac{\mu^2}{m_1\|\mathbf{q_1^\perp}\|^2 + m_2\|\mathbf{q_1^\perp}+\mathbf{q_2^\perp}\|^2 + m_3\|\mathbf{q_1^\perp}+\mathbf{q_2^\perp}+\mathbf{q_3^\perp}\|^2} \end{aligned} \quad (6)$$

3 Relative Equilibria

The interesting equilibria of the system is the relative equilibria of the system. These equilibria characterizes the states of the system as the three spheres rotate uniformly about the z-axis. These equilibria can be computed by determining the critical points of V_μ [1].

The solutions characterizing each pendulum pointing down are first searched for [1]. The constraints for the equilibria being considered are the length constraints on each vector:

$$q_i^3 = -\sqrt{l_i^2 - \|\mathbf{q_i}^\perp\|^2} \qquad \text{for } i = 1, 2, 3$$

[1] the trivial solution $\mathbf{q_i}^\perp = 0$ for $i = 1, 2, 3$ is not being considered

By substituting the above constraints into the amended potential and rearranging, the following is obtained:

$$
\begin{aligned}
V_\mu(\mathbf{q_1}^\perp, \mathbf{q_2}^\perp, \mathbf{q_3}^\perp) \;=\; & -(m_1+m_2+m_3)g\sqrt{l_1^2 - \|\mathbf{q_1}^\perp\|^2} - \\
& (m_2+m_3)g\sqrt{l_2^2 - \|\mathbf{q_2}^\perp\|^2} - m_3 g\sqrt{l_3^2 - \|\mathbf{q_3}^\perp\|^2} + \\
& \frac{\mu^2}{2\mathbb{I}}
\end{aligned} \tag{7}
$$

The critical points are found by setting the derivatives of V_μ with respect to $\mathbf{q_i}^\perp$ to 0 for $i = 1, 2, 3$. The conditions thus obtained are:

$$
(m_1+m_2+m_3)g\frac{\mathbf{q_1}^\perp}{\sqrt{l_1^2 - \|\mathbf{q_1}^\perp\|^2}} = \frac{\mu^2}{\mathbb{I}^2}[(m_1+m_2+m_3)\mathbf{q_1}^\perp + (m_2+m_3)\mathbf{q_2}^\perp + m_3\mathbf{q_3}^\perp] \tag{8}
$$

$$
(m_2+m_3)g\frac{\mathbf{q_2}^\perp}{\sqrt{l_2^2 - \|\mathbf{q_2}^\perp\|^2}} = \frac{\mu^2}{\mathbb{I}^2}[(m_2+m_3)(\mathbf{q_1}^\perp + \mathbf{q_2}^\perp) + m_3\mathbf{q_3}^\perp] \tag{9}
$$

$$
m_3 g\frac{\mathbf{q_3}^\perp}{\sqrt{l_3^2 - \|\mathbf{q_3}^\perp\|^2}} = \frac{\mu^2}{\mathbb{I}^2}[m_3(\mathbf{q_1}^\perp + \mathbf{q_2}^\perp + \mathbf{q_3}^\perp)] \tag{10}
$$

By definition, $\mathbf{q_i}^\perp$ are vectors on the x-y plane, which implies that $\mathbf{q_1}^\perp // \mathbf{q_2}^\perp // \mathbf{q_3}^\perp$. This condition allows the definition of the following parameters:

$$
\begin{aligned}
\mathbf{q_2}^\perp &= \alpha\mathbf{q_1}^\perp \\
\mathbf{q_3}^\perp &= \beta\mathbf{q_1}^\perp \\
\|\mathbf{q_1}^\perp\| &= \lambda l_1
\end{aligned}
$$

The shape of the relative equilibrium can be characterized with α, β, and λ. Furthermore, define the system parameters as follows:

$$
\begin{aligned}
r &= \frac{l_2}{l_1} \\
\hat{r} &= \frac{l_3}{l_1} \\
m &= \frac{m_1+m_2}{m_2} \\
\hat{m} &= \frac{m_3}{m_2}
\end{aligned}
$$

With these definitions, the conditions (7), (8), and (9) can be rewritten as:

$$\frac{(m+\hat{m})g}{\sqrt{1-\lambda^2 l_1}} = \frac{\mu^2}{\mathbb{I}^2}[m+\alpha+\hat{m}(1+\alpha+\beta)] \tag{11}$$

$$\frac{(1+\hat{m})g\alpha}{\sqrt{r^2-\alpha^2\lambda^2 l_1}} = \frac{\mu^2}{\mathbb{I}^2}[1+\alpha+\hat{m}(1+\alpha+\beta)] \tag{12}$$

$$\frac{g\beta}{\sqrt{\hat{r}^2-\beta^2\lambda^2 l_1}} = \frac{\mu^2}{\mathbb{I}^2}(1+\alpha+\beta) \tag{13}$$

The holonomic constraints that $\|\mathbf{q_i}^{\perp}\| \leq l_i$ for $i = 1,2,3$ implies the following constraint on λ:

$$0 \leq \lambda \leq \min\{\frac{r}{\alpha}, \frac{\hat{r}}{\beta}, 1\}$$

Note that if we assume that $\hat{m} = 0$ and $\hat{r} = 0$, that is, the double spherical pendulum setting, we do recover the double pendulum equations as in Marsden and Scheurle [1] [2]. From the above equations, conditions for α and β can be obtained:

$$m+\alpha+\hat{m}(1+\alpha+\beta) > 0 \qquad \text{and}$$

$$\frac{1+\alpha+\hat{m}(1+\alpha+\beta)}{\alpha} > 0 \qquad \text{and}$$

$$\frac{1+\alpha+\beta}{\beta} > 0$$

Dividing equations (11) by (12) and (13) by (11), the following expressions on λ^2 can be obtained:

$$\lambda^2 = \frac{L_1^2 - r^2}{L_1^2 - \alpha^2} \tag{14}$$

$$\lambda^2 = \frac{L_2^2 - \hat{r}^2}{L_2^2 - \beta^2} \tag{15}$$

where

$$L_1^2 = [(\frac{\alpha+\alpha\hat{m}}{m+\hat{m}})\frac{m+\alpha+\hat{m}(1+\alpha+\beta)}{1+\alpha+\hat{m}(1+\alpha+\beta)}]^2$$

$$L_2^2 = [\frac{\beta(m+\alpha+\hat{m}(1+\alpha+\beta))}{(m+\hat{m})(1+\alpha+\beta)}]^2$$

To obtain the values of the shape variables of the system, one realizes that λ^2 has to evaluate to one value from both equations ((14) and (15)), subjected to the constraint $0 \leq \lambda^2 \leq \min\{(\frac{r}{\alpha})^2, (\frac{\hat{r}}{\beta})^2, 1\}$. By equating equation (14) to equation (15) and rearranging, a relation between α and β is obtained:

$$(L_1^2 - r^2)(L_2^2 - \beta^2) - (L_2^2 - \hat{r}^2)(L_1^2 - \alpha^2) = 0 \tag{16}$$

[2] equation (13) (or equivalently equation (10)) drops out since $\hat{m} = 0 \Longleftrightarrow m_3 = 0$

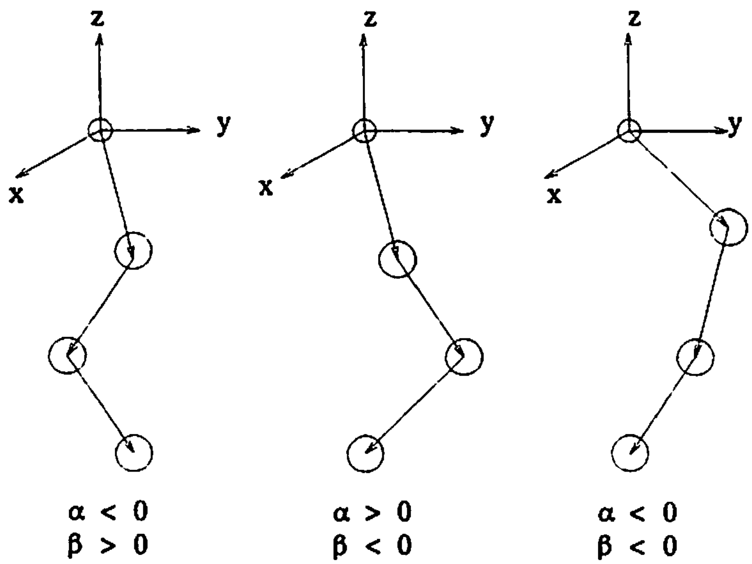

Figure 2: Different possible types of cowboy solutions

In other words, given the physical constraints r, $\hat{r}$, m, and $\hat{m}$, the relative equilibria are defined by solving equation (16) for either α or β and substituting the result in λ^2. Another point worth noting is that there exists three possible kinds of "cowboy" solutions (Figure 2):

- $\alpha < 0$ and $\beta > 0 \Leftrightarrow$ the "true" cowboy solution for the triple spherical pendulum case [3];
- $\alpha > 0$ and $\beta < 0 \Leftrightarrow$ a "pseudo-"cowboy solution: the upper two beads portion acts like a single string with a mass and the system can be thought of as a pseudo-double-spherical-pendulum;
- $\alpha < 0$ and $\beta < 0 \Leftrightarrow$ another "pseudo-"cowboy solution: the lower two beads portion acts like a single string with a mass and the system can be thought of as another pseudo-double-spherical-pendulum.

For a given set of parameters $(r, \hat{r}, m, \hat{m})$, the analysis of possible combinations of α and β will determine the existence of each of these types of cowboy solutions.

For example, for the case of $r = 1$, $\hat{r} = 1$, $m = 2$, and $\hat{m} = 1$, 4 different relative equilibria solutions are obtained and a sample of them is as follows:

- $\alpha = -1.7 \Leftrightarrow \beta = 0.805$ and $\lambda^2 = 0.177$: the first kind of the cowboy solution;

[3]one way to visualize this scenario is to think of the relative equilibrium as a "zig-zag" when projected onto the y-z plane

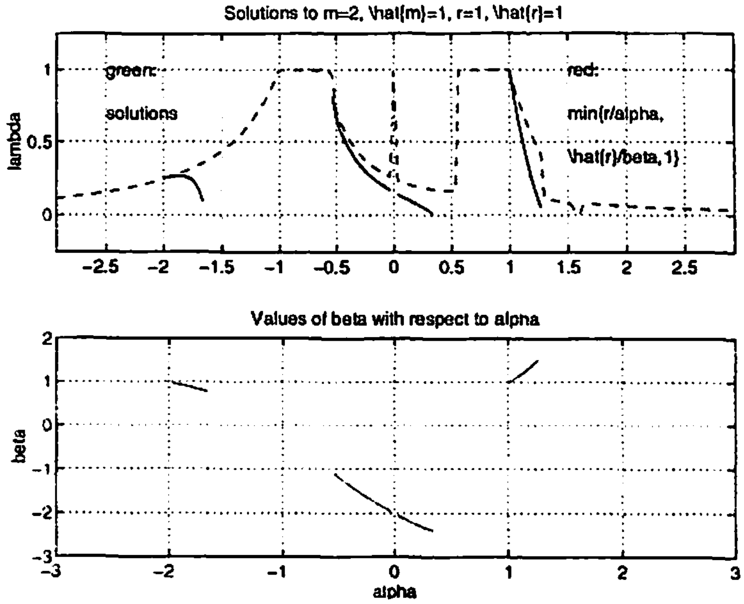

Figure 3: Solutions to $r = 1$, $\hat{r} = 1$, $m = 2$, and $\hat{m} = 1$

- $\alpha = -0.3 \Leftrightarrow \beta = -1.5412$ and $\lambda^2 = 0.345$: the third kind of the cowboy solution:
- $\alpha = 0.3 \Leftrightarrow \beta = -2.354$ and $\lambda^2 = 0.03$: the second kind of the cowboy solution:
- $\alpha = 1.1 \Leftrightarrow \beta = 1.169$ and $\lambda^2 = 0.537$: the trivially rotating solution.

The solutions for $-3 < \alpha < 3$ and $r = 1$, $\hat{r} = 1$, $m = 2$, and $\hat{m} = 1$, are shown in Figure 3. For fixed $m = 2$ and $\hat{m} = 1$, Figure 4 shows the effect of large and small value of the ratio $\frac{r}{\hat{r}}$. So for large $\frac{r}{\hat{r}}$, it appears that the trivial rotation solution does not exist for the specified range of α.

4 Stability of Equilibria

To determine the stability of the relative equilibria of the triple-spherical pendulum system, the idea of the energy momentum method [2] is employed. Polar coordinates (r_i, θ_i) are assigned to $\mathbf{q_i}^{\perp}$ for $i = 1, 2, 3$. Thus, $\phi = \theta_2 - \theta_1$ and $\phi = \theta_3 - \theta_2$ are the S^1-invariant coordinates. By putting the conserved angular momentum J from equation (5) and the Lagrangian L from equation (1) in

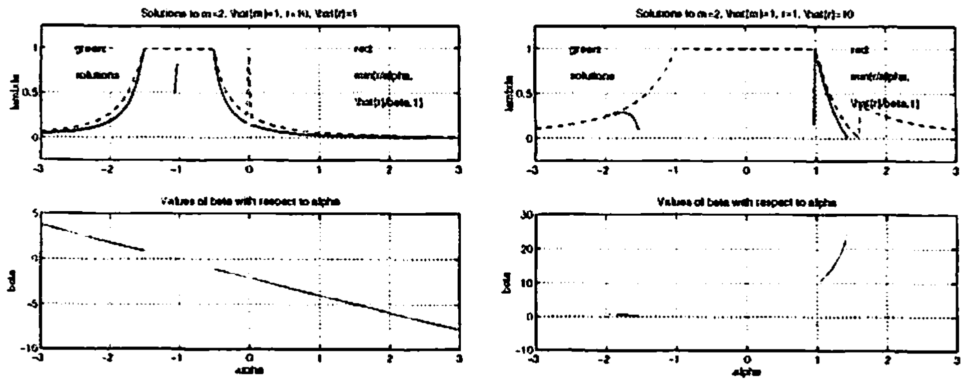

Figure 4: Solutions to $m = 2$ and $\hat{m} = 1$ for large and small $\hat{r}/r$

terms of the polar coordinates:

$$\begin{aligned}
J \;=\; & m_1 r_1^2\dot{\theta}_1 + m_2[r_1^2\dot{\theta}_1 + r_2^2\dot{\theta}_2 + r_1 r_2\dot{\theta}_1\cos\phi + r_1 r_2\dot{\theta}_2\cos\phi - \\
& \dot{r}_1 r_2\sin\phi + r_1\dot{r}_2\sin\phi] + m_3[-\frac{r_3\dot{r}_3}{2} + r_1^2\dot{\theta}_1 + r_2^2\dot{\theta}_2 + \frac{r_3^2\dot{\theta}_3}{2} + \\
& r_1 r_2\dot{\theta}_1\cos\phi + r_1 r_2\dot{\theta}_2\cos\phi + r_1 r_3\dot{\theta}_1\cos(\phi + \hat{\phi}) + \\
& r_1 r_3\dot{\theta}_3\cos(\phi + \hat{\phi}) + r_2 r_3\dot{\theta}_2\cos\hat{\phi} + r_2 r_3\dot{\theta}_3\cos\hat{\phi} + \frac{r_3\dot{r}_3\cos(\phi + \hat{\phi} + \theta_1)}{2} + \\
& \frac{r_3^2\dot{\theta}_3\cos(\phi + \hat{\phi} + \theta_1)}{2} - \dot{r}_1 r_2\sin\phi + r_1\dot{r}_2\sin\phi - \dot{r}_1 r_3\sin(\phi + \hat{\phi}) + \\
& r_1\dot{r}_3\sin(\phi + \hat{\phi}) - \dot{r}_2 r_3\sin\hat{\phi} + r_2\dot{r}_3\sin\hat{\phi} + \frac{r_3\dot{r}_3\sin(\phi + \hat{\phi} + \theta_1)}{2} - \\
& \frac{r_3^2\dot{\theta}_3\sin(\phi + \hat{\phi} + \theta_1)}{2}] \qquad (17)
\end{aligned}$$

$$\begin{aligned}
L \;=\; & \frac{1}{2}m_1[\dot{r}_1^2 + r_1^2\dot{\theta}_1^2 + \frac{r_1^2\dot{r}_1^2}{l_1^2 - r_1^2} + g\sqrt{l_1^2 - r_1^2}] + \frac{1}{2}m_2[\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + \\
& 2\dot{r}_1\dot{r}_2\cos\phi + 2r_1 r_2\dot{\theta}_1\dot{\theta}_2\cos\phi + 2r_1\dot{r}_2\dot{\theta}_1\sin\phi - 2\dot{r}_1 r_2\dot{\theta}_2\sin\phi + \\
& (\frac{r_1\dot{r}_1}{\sqrt{l_1^2 - r_1^2}} + \frac{r_2\dot{r}_2}{\sqrt{l_2^2 - r_2^2}})^2 + g(\sqrt{l_1^2 - r_1^2}\sqrt{l_2^2 - r_2^2})] + \frac{1}{2}m_3[\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2 + \\
& r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + r_3^2\dot{\theta}_3^2 + 2\dot{r}_1\dot{r}_2\cos\phi + 2r_1 r_2\dot{\theta}_1\dot{\theta}_2\cos\phi + 2\dot{r}_1\dot{r}_3\cos(\phi + \hat{\phi}) + \\
& 2r_1 r_3\dot{\theta}_1\dot{\theta}_3\cos\hat{\phi} + 2\dot{r}_2\dot{r}_3\cos\hat{\phi} + 2r_2 r_3\dot{\theta}_2\dot{\theta}_3\cos\hat{\phi} + \\
& 2r_1\dot{r}_2\dot{\theta}_1\sin\phi - 2\dot{r}_1 r_2\dot{\theta}_2\sin\phi + 2r_1\dot{r}_3\dot{\theta}_1\sin(\phi + \hat{\phi}) - \\
& 2\dot{r}_1 r_3\dot{\theta}_3\sin(\phi + \hat{\phi}) + 2r_2\dot{r}_3\dot{\theta}_2\sin\hat{\phi} - 2\dot{r}_2 r_3\dot{\theta}_3\sin\hat{\phi} + \\
& (\frac{r_1\dot{r}_1}{\sqrt{l_1^2 - r_1^2}} + \frac{r_2\dot{r}_2}{\sqrt{l_2^2 - r_2^2}} + \frac{r_3\dot{r}_3}{\sqrt{l_3^2 - r_3^2}})^2 + \\
& g(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2} + \sqrt{l_3^2 - r_3^2})] \qquad (18)
\end{aligned}$$

the amended potential V_μ can be rewritten in the following form:

$$\begin{aligned} V_\mu &= -m_1 g\sqrt{l_1^2 - r_1^2} - m_2 g(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2}) - \\ &\quad m_3 g(\sqrt{l_1^2 - r_1^2} + \sqrt{l_1^2 - r_1^2} + \sqrt{l_3^2 - r_3^2}) + \frac{1}{2}\frac{\mu^2}{\hat{\mathbb{I}}} \end{aligned} \tag{19}$$

where

$$\begin{aligned} \hat{\mathbb{I}} &= m_1 r_1^2 + m_2(r_1^2 + r_2^2 + 2r_1 r_2 \cos\phi) + \\ &\quad m_3[r_1^2 + r_2^2 + r_3^2 + 2r_1 r_2 \cos\phi + 2r_1 r_3 \cos(\phi + \hat{\phi}) + \\ &\quad 2r_2 r_3 \cos\hat{\phi}] \end{aligned}$$

If m_3 is set to zero, the double spherical pendulum case is again recovered. One way to determine the stability of the relative equilibria of the system is to calculate $\delta^2 V_\mu$. After some algebra, it can be obtained as follows:

$$\delta^2 V_\mu = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ b_1 & a_2 & c_2 & 0 & 0 \\ c_1 & c_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & e_4 \\ 0 & 0 & 0 & e_4 & a_5 \end{bmatrix} \tag{20}$$

where

$$\begin{aligned} a_1 &= \frac{\mu^2[4(\alpha + m + \hat{m} + \alpha\hat{m} + \beta\hat{m})^2]}{\lambda^4 l_1^4 m_2(2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m})^3} + \\ &\quad \frac{\mu^2(-m - \hat{m})}{\lambda^4 l_1^4 m_2(2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m})^2} + \\ &\quad \frac{g m_2(m + \hat{m})}{\sqrt{l_1^2(1 - \lambda^2)}} + \frac{g\lambda^2 l_1^2 m_2(m + \hat{m})}{(l_1^2(1 - \lambda^2))^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{\mu^2[4(1 + \alpha + \hat{m} + \alpha\hat{m} + \beta\hat{m})(\alpha + m + \hat{m} + \alpha\hat{m} + \beta\hat{m})]}{\lambda^4 l_1^4 m_2(2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m})^3} + \\ &\quad \frac{\mu^2(-1 - \hat{m})}{\lambda^4 l_1^4 m_2(2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m})^2} \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{\mu^2[4(1 + \alpha + \beta)\hat{m}(\alpha + m + \hat{m} + \alpha\hat{m} + \beta\hat{m}) - \hat{m}]}{\lambda^4 l_1^4 m_2(2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m})^3} + \\ &\quad \frac{-\mu^2\hat{m}}{\lambda^4 l_1^4 m_2(2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m})^2} \end{aligned}$$

$$
\begin{aligned}
a_2 &= \frac{\mu^2[4(1+\alpha+\hat{m}+\alpha\hat{m}+\beta\hat{m})^2]}{\lambda^4 l_1^4 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^3}+\\
&\quad \frac{\mu^2(-1-\hat{m})}{\lambda^4 l_1^4 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^2}+\\
&\quad \frac{g m_2(1+\hat{m})}{\sqrt{l_1^2(r^2-\alpha^2\lambda^2)}}+\frac{g\lambda^2 l_1^2 m_2\alpha^2(1+\hat{m})}{(l_1^2(r^2-\alpha^2\lambda^2))^{\frac{3}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= \frac{\mu^2[4(1+\alpha+\beta)\hat{m}(1+\alpha+\hat{m}+\alpha\hat{m}+\beta\hat{m})]}{\lambda^4 l_1^4 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^3}+\\
&\quad \frac{-\mu^2\hat{m}}{\lambda^4 l_1^4 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^2}
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= \frac{\mu^2[4\hat{m}^2(1+\alpha+\beta)^2]}{\lambda^4 l_1^4 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^3}+\\
&\quad \frac{-\mu^2\hat{m}}{\lambda^4 l_1^4 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^2}+\\
&\quad \frac{g m_2\hat{m}}{\sqrt{l_1^2(\hat{r}^2-\beta^2\lambda^2)}}+\frac{g\lambda^2 l_1^2 m_2\hat{m}\beta^2}{(l_1^2(\hat{r}^2-\beta^2\lambda^2))^{\frac{3}{2}}}
\end{aligned}
$$

$$
a_4 = \frac{(\alpha+\alpha\hat{m}+\beta\hat{m})\mu^2}{\lambda^2 l_1^2 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^2}
$$

$$
e_4 = \frac{\beta\hat{m}\mu^2}{\lambda^2 l_1^2 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^2}
$$

$$
a_5 = \frac{(1+\alpha)\beta\hat{m}\mu^2}{\lambda^2 l_1^2 m_2(2\alpha+\alpha^2+m+\hat{m}+2\alpha\hat{m}+\alpha^2\hat{m}+2\beta\hat{m}+2\alpha\beta\hat{m}+\beta^2\hat{m})^2}
$$

As explained in Marsden and Scheurle [1], the signature of $\delta^2 V_\mu$ determines the stability of the relative equilibrium of interest. For the case of $r=1$, $\hat{r}=1$, $m=2$, and $\hat{m}=1$, the signature of $\delta^2 V_\mu$ is as follows:

$$
\begin{aligned}
-2<\alpha<-1 &\iff (-,-,-,-,-)\\
-1<\alpha<0 &\iff (-,-,-,-,-)\\
0<\alpha<1 &\iff (-,-,+,-,-)\\
1<\alpha<2 &\iff (+,+,+,+,+)
\end{aligned}
$$

So only the trivial rotation and the second kind of cowboy solution are stable.

5 Linearized Equations of Motion and Bifurcation Analysis

The Euler-Lagrange equations of motion can be obtained for $(r_1, r_2, r_3, \theta_1, \theta_2, \theta_3)$ from the Lagrangian described in equation (18). The resulting equations can then be dropped down to $J^{-1}(\mu)/G_\mu$ where $G_\mu = G =$ Lie group for Abelian groups. This operation amounts to rewriting the Euler-Lagrange equations in terms of ϕ and $\hat{\phi}$, as defined before, using the equivariant momentum map J (which is set to μ) described in equation (17). The resulting equations can then be linearized about a given relative equilibrium solution to determine its stability. The linearized equations should take the form:

$$M\ddot{x} + S\dot{x} + \Lambda x = 0 \tag{21}$$

where

$$\begin{aligned}
x &= (r_1, r_2, r_3, \phi, \hat{\phi}) \\
M &= \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 \\ m_{12} & m_{22} & m_{23} & 0 & 0 \\ m_{13} & m_{23} & m_{33} & 0 & 0 \\ 0 & 0 & 0 & m_{44} & m_{45} \\ 0 & 0 & 0 & m_{45} & m_{55} \end{bmatrix} \\
S &= \begin{bmatrix} 0 & 0 & 0 & s_{14} & s_{15} \\ 0 & 0 & 0 & s_{24} & s_{25} \\ 0 & 0 & 0 & s_{34} & s_{35} \\ -s_{14} & -s_{24} & -s_{34} & 0 & 0 \\ -s_{15} & -s_{25} & -s_{35} & 0 & 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
m_{11} &= \frac{(m_1 + m_2 + m_3)}{1 - \lambda^2} \\
m_{12} &= (m_2 + m_3)(1 + \frac{\alpha\lambda^2}{\sqrt{(1-\lambda^2)}\sqrt{(r^2 - \alpha^2\lambda^2)}}) \\
m_{13} &= m_3(1 + \frac{\beta\lambda^2}{\sqrt{(1-\lambda^2)}\sqrt{(\hat{r}^2 - \beta^2\lambda^2)}}) \\
m_{22} &= \frac{(m_2 + m_3)r^2}{r^2 - \alpha^2\lambda^2} \\
m_{23} &= m_3(1 + \frac{\alpha\beta\lambda^2}{\sqrt{(r^2 - \alpha^2\lambda^2)}\sqrt{(\hat{r}^2 - \beta^2\lambda^2)}}) \\
m_{33} &= \frac{m_3 r^2}{\hat{r}^2 - \beta^2\lambda^2}
\end{aligned}$$

$$
\begin{aligned}
m_{44} &= \frac{\lambda^2 l_1^2 m_2 (m-1)(\alpha^2 + \hat{m}\alpha^2 + \alpha\beta\hat{m})}{2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{b} + 2\alpha\beta\hat{m} + \beta^2\hat{m}} \\
m_{45} &= \frac{\hat{m} m_2 \alpha\beta\lambda^2 l_1^2 (m-1)}{2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m}} \\
m_{55} &= \frac{\hat{m} m_2 \alpha\beta^2\lambda^2 l_1^2 (\alpha+1)}{2\alpha + \alpha^2 + m + \hat{m} + 2\alpha\hat{m} + \alpha^2\hat{m} + 2\beta\hat{m} + 2\alpha\beta\hat{m} + \beta^2\hat{m}}
\end{aligned}
$$

and Λ is the matrix given by equation (20). At the time of writing this report, the author has not finished the algebra needed in computing the matrix S, though it should be skew-symmetric in theory [1]. Again, note that the matrices M and Λ reduce to the case of the double-spherical pendulum when m_3 is set to zero, and S should also, in theory.

Given the linearized system described in equation (21), the characteristic polynomial $p(\gamma)$ (with eigenvalue γ) can be obtained as:

$$p(\gamma) = \det[\gamma^2 M + \gamma S + \Lambda] \tag{22}$$

Given the various relevant parameters (r, $\hat{r}$, m, $\hat{m}$, α, β, and μ), the linearized stability of a relative equilibrium solution can then be determined, and bifurcations can be observed as the changes in the parameters alter the nature of the eigenvalues.

6 Discussions and Further Work

In this report, the triple-spherical pendulum is considered. The relative equilibria are obtained and their stability analyzed using the concept of the energy-momentum method [2] with an amended potential. In an attempt to characterize the bifurcations behavior of a relative equilibrium of interest, the equations of motion are linearized in terms of the S^1-invariant angles. A computational problem in obtaining the matrix associated with the magnetic term is encountered and this particular matrix is not completely computed at the time of preparation of this report, though the steps for computing this matrix are outlined in the previous section.

Despite the problem mentioned above, some interesting behavior is observed with the triple-spherical pendulum case:

- there exists two bifurcation parameters (α and β) as opposed to one (α) in the double-spherical pendulum case;
- the existence and nature of relative equilibrium solutions depends upon the physical parameters (r, $\hat{r}$, m, $\hat{m}$) of the system and only exist for certain combinations of values of α and β;
- there exist three distinct kinds of "cowboy" solutions; for the case of $r = 1$, $\hat{r} = 1$, $m = 2$, and $\hat{m} = 1$, the only stable solutions are the trivial rotation and the cowboy solution that resembles the double-spherical pendulum

case where the upper two beads portion acts like a single string with a mass.

The first and the third items suggest that there are certain similarities between the double- and the triple-spherical pendulum cases. The analysis technique applies in both cases, and the results are similar [4].

Appendix

By assigning polar coordinates (r_i, θ_i) to $\mathbf{q_i}^{\perp}$ for $i = 1, 2, 3$, the following relationships can be obtained and used to rewrite equations (1), (5), and (6) as (17), (18), and (19) respectively.

$$
\begin{aligned}
\mathbf{k} \cdot [\mathbf{q}_1 \times \dot{\mathbf{q}}_1] &= r_1^2\dot{\theta}_1 \\
\mathbf{k} \cdot [(\mathbf{q}_1 + \mathbf{q}_2) \times (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2)] &= r_1^2\dot{\theta}_1 + r_2^2\dot{\theta}_2 + r_1 r_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \\
&\quad r_1 r_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1) - \dot{r}_1 r_2 \sin(\theta_2 - \theta_1) + \\
&\quad r_1 \dot{r}_2 \sin(\theta_2 - \theta_1) \\
\mathbf{k} \cdot [(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \times (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3)] &= -\frac{r_3\dot{r}_3}{2} + r_1^2\dot{\theta}_1 + r_2^2\dot{\theta}_2 + \frac{r_3^2\dot{\theta}_3}{2} + \\
&\quad r_1 r_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + r_1 r_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \\
&\quad r_1 r_3 \dot{\theta}_1 \cos(\theta_3 - \theta_1) + r_1 r_3 \dot{\theta}_3 \cos(\theta_3 - \theta_1) + \\
&\quad r_2 r_3 \dot{\theta}_2 \cos(\theta_3 - \theta_2) + r_2 r_3 \dot{\theta}_3 \cos(\theta_3 - \theta_2) + \\
&\quad \frac{r_3\dot{r}_3 \cos(2\theta_3)}{2} + \frac{r_3^2\dot{\theta}_3 \cos(2\theta_3)}{2} - \\
&\quad \dot{r}_1 r_2 \sin(\theta_2 - \theta_1) + r_1 \dot{r}_2 \sin(\theta_2 - \theta_1) - \\
&\quad \dot{r}_1 r_3 \sin(\theta_3 - \theta_1) + r_1 \dot{r}_3 \sin(\theta_3 - \theta_1) - \\
&\quad \dot{r}_2 r_3 \sin(\theta_3 - \theta_2) + r_2 \dot{r}_3 \sin(\theta_3 - \theta_2) + \\
&\quad \frac{r_3\dot{r}_3 \sin(2\theta_3)}{2} - \frac{r_3\dot{r}_3 \sin(2\theta_3)}{2} \\
\|\dot{\mathbf{q}}_1\|^2 &= \dot{r}_1^2 + r_1^2\dot{\theta}_1^2 + \frac{r_1^2\dot{r}_1^2}{l_1^2 - r_1^2} \\
\|\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2\|^2 &= \dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + \\
&\quad 2\dot{r}_1\dot{r}_2 \cos(\theta_2 - \theta_1) + 2 r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \\
&\quad 2 r_1 \dot{r}_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1) - 2\dot{r}_1 r_2 \dot{\theta}_2 \sin(\theta_2 - \theta_1) + \\
&\quad \left(\frac{r_1\dot{r}_1}{\sqrt{l_1^2 - r_1^2}} + \frac{r_2\dot{r}_2}{\sqrt{l_2^2 - r_2^2}}\right)^2
\end{aligned}
$$

[4] in fact, given a clever enough coding scheme with a software package capable of symbolic mathematical manipulations (such as Mathematica or Maple), the case of n-tuple pendulum can be studied where n is completely arbitrary; the inputs to the software can be the physical constants m_i and l_i for $i = 1, 2, \ldots, n$ and the outputs are the bifurcation/shape parameters, possible solutions, stability results with $\delta^2 V_\mu$, linearized equations of motion on $J^{-1}(\mu)/G_\mu$, and the bifurcations behavior based on the nature of the eigenvalues

$$
\begin{aligned}
\|\dot{q}_1+\dot{q}_2+\dot{q}_3\|^2 &= \dot{r}_1^2+\dot{r}_2^2+\dot{r}_3^2+r_1^2\dot{\theta}_1^2+r_2^2\dot{\theta}_2^2+\\
&\quad r_3^2\dot{\theta}_3^2+2\dot{r}_1\dot{r}_2\cos(\theta_2-\theta_1)+\\
&\quad 2r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)+2\dot{r}_1\dot{r}_3\cos(\theta_3-\theta_1)+\\
&\quad 2r_1r_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_3-\theta_1)+2\dot{r}_2\dot{r}_3\cos(\theta_3-\theta_2)+\\
&\quad 2r_2r_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_3-\theta_2)+2r_1\dot{r}_2\dot{\theta}_1\sin(\theta_2-\theta_1)-\\
&\quad 2\dot{r}_1r_2\dot{\theta}_2\sin(\theta_2-\theta_1)+2r_1\dot{r}_3\dot{\theta}_1\sin(\theta_3-\theta_1)-\\
&\quad 2\dot{r}_1r_3\dot{\theta}_3\sin(\theta_3-\theta_1)+2r_2\dot{r}_3\dot{\theta}_2\sin(\theta_3-\theta_2)-\\
&\quad 2\dot{r}_2r_3\dot{\theta}_3\sin(\theta_3-\theta_2)+(\frac{r_1\dot{r}_1}{\sqrt{l_1^2-r_1^2}}+\\
&\quad \frac{r_2\dot{r}_2}{\sqrt{l_2^2-r_2^2}})+\frac{r_3\dot{r}_3}{\sqrt{l_3^2-r_3^2}})^2\\
\|\dot{q}_1^\perp\|^2 &= r_1^2\\
\|\dot{q}_1^\perp+\dot{q}_2^\perp\|^2 &= r_1^2+r_2^2+2r_1r_2\cos(\theta_2-\theta_1)\\
\|\dot{q}_1^\perp+\dot{q}_2^\perp+\dot{q}_3^\perp\|^2 &= r_1^2+r_2^2+r_3^2+2r_1r_2\cos(\theta_2-\theta_1)+\\
&\quad 2r_1r_3\cos(\theta_3-\theta_1)+2r_2r_3\cos(\theta_3-\theta_2)
\end{aligned}
$$

References

[1] J.E. Marsden and J. Scheurle. Lagrangian reductionand the double spherical pendulum. *Zeitschrift Fur Angenwandte Mathematik und Physik*, 44(1):17–43, Jan 1993.

[2] J.C. Simo, D. Lewis, and J.E. Marsden. Stability of relative equilibria i: The reduced energy momentum method. *Archive for Rational Mechanics and Analysis*, 115(1):15–59, 1991.