Nonholonomic Mechanical Systems

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Jeff - this is an excellent report! A+
J.M.

1 Introduction

This paper describes work on nonholonomic systems with and without symmetry. This report is based essentially on three papers. Various books were consulted for the necessary background. The three papers are Bates and Sniatycki[1]; Bloch, Krishnaprasad, Marsden, and Murray[2]: and van der Schaft and Mascheke[6]. The following texts were frequently consulted for background: Kobayashi and Nomizu[3], Marsden and Ratiu[4], and Spivak[5]. I began reading Bloch, Krishnaprasad, Marsden, and Murray[2] which is in preparation and have included some of the results of that paper here. I read the Bates and Sniatycki[1] paper but do not include directly the results in this report. I present most of the results in van der Schaft and Mascheke[6] with additional and different explanations for the statements in the paper. One of the goals of this report, besides learning more about this subject, was to examine the pseudo-Poisson bracket. The bracket for these systems satisfies the first two identities for the Poisson bracket but fails the Jacobi identity. This report sets out to further understand the remainder of the Jacobi identity calculation.

The results in [2] and [6] are demonstrated in the vertical penny rolling on a horizontal plane. The remainder in the Jacobi equation is calculated for this example and is shown to be an equation involving the curvature of an Ehresmann connection defined by the nonholonomic constraints. The report concludes with a conjecture for the remainder in the Jacobi identity calculation that involves the curvature of the Ehresmann connection. On the pseudo-symplectic side, Bates and Sniatycki[1] calculate a formula for the remainder of the exterior derivative for nonholonomic systems and relate it to the curvature of a principal connection. Their two-form defines a corresponding bracket. The fact that their two form is not closed implies that the bracket does not satisfy Jacobi's identity. This report sets out to conjecture a remainder equation for Jacobi's identity for nonholonomic systems with Ehresmann connections determined by the constraints. The conjectured equation resembles the equation in Bates and Sniatycki[1].

The first section presents background from Bloch, Krishnaprasad, Marsden, and Murray[2] on principal and Ehresmann connections. The *momentum equation*, a result from their paper, is then presented. An exposition is then given on van der Schaft and Mascheke[6] which calculates the equations on the constraint phase space followed by a formula for the bracket tensor. A theorem is presented and proved showing that the bracket satisfies the Jacobi identity if and only if the constraints are holonomic. The paper concludes by analyzing the vertical penny rolling on a horizontal plane using the results in Bloch, Krishnaprasad, Marsden, and Murray[2] and in van der Schaft and Mascheke[6]. The *momentum equation* is described for this example and is used to solve the equations of motion. Following the development in van der Schaft and Mascheke[6], the equations of motion on the constraint phase space are derived. The bracket and the remainder in Jacobi's identity is calculated and related to the curvature of the Ehresmann connection determined by the constraints. The paper finally presents a conjectured equation for the remainder of the Jacobi identity calculation.

2 A Few Results and Definitions from "Nonholonomic Mechanical Systems with Symmetry"

In this section, a subset of the results from "Nonholonomic Mechanical Systems with Symmetry" are presented. Definitions and results from connection theory are discussed which are needed for subsequent sections of the report. The report then develops the momentum equation which arrives from consideration of the group invariance of the Lagrangian and the nonholonomic constraints. In a later section, these results are used in an example.

2.1 Principal and Ehresmann Connections

In this report, connections provide a method of characterizing the nonholonomic constraints. The allowable directions are given by the horizontal space of the connection. The results presented in class on principal connections are given here. Some definitions and results of Ehresmann connections are given which are used later in the report.

Let Q be a manifold and let a group G act on Q on the left. Let the action of G be free and proper so that the quotient space, Q/G, is a manifold. The manifold Q is a principal bundle with the projection $\pi : Q \rightarrow Q/G$. A *(principal) connection* on the bundle is a map from the tangent space to the lie algebra and is denoted, $\mathfrak{A} : TQ \rightarrow \mathfrak{g}$. The connection is linear on each tangent space and is therefore a $\mathfrak{g}$-valued 1-form. The connection satisfies these properties:

1. $\mathfrak{A}(\xi_Q(q)) = \xi$ for all $\xi \in \mathfrak{g}$ and $q \in Q$, and
2. $\mathfrak{A}(T_q\Phi_g(v_q)) = \mathrm{Ad}_g\mathfrak{A}(v_q)$ for all $v_q \in T_qQ$ and $g \in G$, where Φ_g is the given action of G on Q.

The vertical space at a point $q \in Q$ is the kernel of $T_q\pi$ and is denoted ver_q. The vertical projection of a tangent vector is

$$\mathrm{ver}_q v_q = \left(\mathfrak{A}\left(v_q\right)\right)_Q(q)$$

The horizontal space at a point $q \in Q$ is

$$\mathrm{hor}_q = \{v_q \in T_qQ | \mathfrak{A}(v_q) = 0\}$$

The curvature of the connection is a $\mathfrak{g}$-valued form given by the exterior derivative of the connection acting on horizontal projections of the vectors and is denoted

$$\mathfrak{B}(X,Y) = d\mathfrak{A}(\mathrm{hor}X, \mathrm{hor}Y). \tag{1}$$

As presented in class, there are other forms of the curvature that are sometimes easier to use than (1). Two equivalent equations for the curvature are

$$\mathfrak{B}(X,Y) = -\mathfrak{A}\left([\mathrm{hor}X, \mathrm{hor}Y]\right) = d\mathfrak{A}(X,Y) - [\mathfrak{A}(X), \mathfrak{A}(Y)] \tag{2}$$

where the first bracket is the Jacob-Lie bracket of vector fields and the second bracket is the Lie algebra bracket.

Ehresmann connections are now described. The Ehresmann connection provides the ability to split T_qQ into horizontal parts and vertical parts as in the principal case; however, the Ehresmann connection is a vertical-valued form and does not require a principal bundle. Given a bundle $\pi : Q \to R$ where Q is the configuration manifold and R is called the base space, the Ehresmann connection maps a tangent vector in T_qQ to its vertical component. As before, the vertical space is the kernel of $T_q\pi$ and the horizontal space is the kernel of $A(q)$ at a point $q \in Q$. It is assumed that there are coordinates for the bundle (r^α, s^a) where r^α are coordinates for R and the remaining coordinates s^a are the fiber coordinates. In coordinates, the projection π acting on a point in Q returns the r^α coordinates. The connection can be represented as a vector valued form ω^a where

$$\omega^a = ds^a + A^a_\alpha(r,s)\, dr^\alpha. \tag{3}$$

Notice that A^a_α is a function of the base and the fiber variables. In a later section of this report, the vertical penny rolling on a plane is examined. In this case, A^a_α is only a function of the base variables. The snake board is an example where the coefficients are functions of the fiber coordinates. The vertical projection in local coordinates is given by

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (0, \dot{s}^a + A^a_\alpha(r,s)\,\dot{r}^\alpha),$$

and the horizontal projection is given by

$$(\dot{r}^{\alpha}, \dot{s}^{a}) \mapsto (\dot{r}^{\alpha}, -A^{a}_{\alpha}(r,s)\,\dot{r}^{\alpha}).$$

The constraint forms for nonholonomic constraints are often given in the form of (3). In this case, a vector satisfies the constraints if and only if it is horizontal. The curvature of the Ehresmann connection is given by

$$B(X_q, Y_q) = -A(q)([\mathrm{hor}X, \mathrm{hor}Y]) \tag{4}$$

where $X_q, Y_q \in T_qQ$ and the vector fields on the right are horizontal extensions of the vectors. It is not clear that this definition is independent of the extension. If there exists a relationship similar to the second equation in (2), then the curvature is independent of the extension. The exterior derivative is not defined for an Ehresmann connection since it is not a mapping from the tangent space to the reals or to a vector space. In local coordinates, one can show that the curvature is given by (1) where the exterior derivative of the local form (3) is taken, and the bracket is the Jacobi-Lie bracket. This shows that it is independent of the extension and gives a useful formula for calculating the curvature of the Ehresmann connection. In the penny example, the curvature is calculated by taking the exterior derivative of the local form of the constraints and applying it to horizontal vectors.

2.2 Momentum Equation

The momentum equation is now described which is a result developed in "Nonholonomic Mechanical Systems with Symmetry" that applies to nonholonomic systems with symmetry. This development begins with a proof of Noether's Theorem and then uses an analogous procedure to develop the momentum equation. This development follows closely the presentation in "Nonholonomic Mechanical Systems with Symmetry".

For the following development, there is a Lie group G acting on a configuration manifold Q and the action will be denoted $q \mapsto gq = \Phi_g(q)$. The nonholonomic kinematic constraints define a distribution $D_q \subset T_qQ$ where D_q is the distribution of *allowable directions*, i.e. in the kernel of the constraint forms. It is also assumed that D_q is invariant to the group action. In other words,

$$T_q\Phi_g \cdot D_q = D_{gq}.$$

Given a configuration manifold and a G-invariant Lagrangian $L : TQ \to \mathbb{R}$, the corresponding conserved momentum map is given by the mapping $\mathbf{J} : TQ \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle \tag{5}$$

where $\mathbb{F}L$ is the fiber derivative. In coordinates,

$$J_a = \frac{\partial L}{\partial \dot{q}^i} K^i_a \tag{6}$$

is the momentum equation where K_a^i are the action coefficients given by $\xi_Q(q^i) = K_a^i \xi^a \partial/\partial \dot{q}^i$. Noether's theorem states that the momentum given by (5) or (6) is constant in time for solutions to the Euler-Lagrange equations. The "Nonholonomic Mechanical Systems with Symmetry" paper presents a proof of Noether's theorem which is restated here. Choose any function $\phi(t,s)$ of two variables such that $\phi(a,s) = \phi(b,s) = \phi(t,0) = 0$ where a and b are endpoints of the given solution of the Euler-Lagrange's equations. Since L is G-invariant, the expression

$$\int_a^b L\left(\exp\left(\phi(t,s)\xi\right)\cdot q, \exp\left(\phi(t,s)\xi\right)\cdot \dot{q}\right) dt \tag{7}$$

is independent of s for each Lie algebra element $\xi \in \mathfrak{g}$. The item in the second argument of the Lagrangian is the tangent map of the group action acting on a tangent vector. Differentiating this expression with respect to s and setting $s = 0$, gives

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i}\xi_Q^i \phi' + \frac{\partial L}{\partial \dot{q}^i}\left(T\xi_Q \cdot \dot{q}\right)^i \phi'\right) dt. \tag{8}$$

Consider the variation $q(t,s) = \exp\left(\phi(t,s)\xi\right)\cdot q(t)$. The corresponding variation (derivative with respect to s) is $\delta q(t) = \phi'(t)\,\xi_Q(q(t))$. Hamilton's principle states that

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i\right) dt. \tag{9}$$

The time derivative of the variation with respect to t is given by

$$\delta \dot{q} = \dot{\phi}'\xi_Q + \phi'\left(T\xi_Q \cdot \dot{q}\right). \tag{10}$$

Substitute (10) into (9) and subtract (8) from the result to get

$$\begin{aligned} 0 &= \int_a^b \frac{\partial L}{\partial \dot{q}^i}\left(\xi_Q\right)^i \dot{\phi}' dt \\ &= -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\xi_Q^i\right)\phi' dt. \end{aligned} \tag{11}$$

The last result follows from integration by parts. Since ϕ' is arbitrary, except for the endpoint conditions, the integrand is zero, and, therefore, the time derivative of the momentum map is zero.

The authors of "Nonholonomic Mechanical Systems with Symmetry" now introduce additional notation for the development of the momentum equation. Define the vertical subbundle Vert of TQ, with the fiber at q given by

$$\left(\text{Vert}\right)_q = \text{span}\left(\xi_Q | \xi \in \mathfrak{g}\right).$$

Now define D_{Vert} to be the union over q of $D_q \cap (\text{Vert})_q$. Now consider a section of D_{Vert}, *i.e.*, a mapping that takes q to an element of $\left(D_{\text{Vert}}\right)_q$. An element in the image of this map is denoted ξ^q_Q. The corresponding Lie algebra element ξ^q is the element of $\mathfrak{g}$ that gives rise to the infinitesimal generator ξ^q_Q.

Define variations analogous to the variations in the proof of Noether's theorem where ξ is replaced with ξ^q, *i.e.*, $q(t,s) = \exp\left(\phi(t,s)\,\xi^{q(t)}\right) \cdot q(t)$. The infinitesimal variation is then $\delta q(t) = \phi'(t)\,\xi^q_Q(q(t))$. Let $\partial\xi^q$ denote the derivative of ξ^q with respect to q so that

$$\delta\dot{q} = \dot{\phi}'\xi^{q(t)}_Q + \phi'\left[\left(T\xi^{q(t)}_Q \cdot \dot{q}\right) + \left(\partial\xi^{q(t)} \cdot \dot{q}\right)_Q\right]. \tag{12}$$

The variations satisfy the constraints and the curve $q(t)$ satisfies the Euler-Lagrange equations so that Hamilton's principle holds given in equation (9). Also, equation (8) holds with ξ replaced with $\xi^{q(t)}$:

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i}\left(\xi^{q(t)}_Q\right)^i \phi' + \frac{\partial L}{\partial \dot{q}^i}\left(T\xi^{q(t)}_Q \cdot \dot{q}\right)^i \phi'\right) dt. \tag{13}$$

Using (12) in (9) and subtracting (13) from the result gives

Theorem 2.1 *Using the notation introduced above, any solution of the Lagrange-d'Alembert equations for a nonholonomic system must satisfy, in addition to the given kinematic constraints, the* ***momentum equation:***

$$\frac{d}{dt}\left(J^c\left(\xi^{q(t)}\right)\right) = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}\left(\xi^{q(t)}\right)\right]^i_Q$$

where $J^c\left(\xi^{q(t)}\right) = \frac{\partial L}{\partial \dot{q}^i}\left(\xi^{q(t)}\right)^i_Q$ *is the* ***nonholonomic momentum map.***

The use of this equation is demonstrated in the rolling vertical penny example in a following section.

3 Exposition on "On the Hamiltonian Formulation of Nonholonomic Mechanical Systems"

In this section, most of the results in "On the Hamiltonian Formulation of Nonholonomic Mechanical Systems" are presented. The nonholonomic constraints in terms of a matrix equation are described and related to the notation above. The Lagrange-d'Alembert equations of motion with Lagrange multipliers are then presented. This equation is then translated to Hamilton's equations with multipliers. Through a specific choice of coordinates, the multipliers are eliminated giving an equation on the constrained phase space. The resulting equations possess a bracket that shares two properties of the Poisson bracket but does not necessarily satisfy Jacobi's identity. The main result of the paper is that the resulting bracket satisfies Jacobi's identity if and only if the constraints are holonomic.

3.1 Hamilton's Equations with Constraints

As before, Q is the configuration manifold with local coordinates $q = (q_1, q_2, \cdots, q_n)$. The Lagrangian is assumed to be regular. The k constraints which are assumed to be linear on the velocities are given by

$$A^T(q)\,\dot{q} = 0 \tag{14}$$

where A^T is a $k \times n, k \leq n$ matrix and is assumed to have rank equal to k for every configuration. The rows of A^T (the columns of A) define the constraint 1-forms. The distribution of allowable directions , D, is given by the kernel of $A^T(q)$. The constraints are *holonomic* if the distribution is involutive ($X, Y \in D \Rightarrow [X, Y] \in D$) by Frobenius' theorem. The constraints are *nonholonomic* if the distribution is not involutive.

The equations of motion for the mechanical system are given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} &= A(q)\,\lambda \\ A^T(q)\,\dot{q} &= 0 \end{aligned} \tag{15}$$

where the derivatives are column vectors and $\lambda(t) \in \mathbb{R}^k$.

The Hamiltonian is defined by the Legendre transformation of the Lagrangian to give

$$H(q, p) = p_i q^i - L(q, \dot{q}), \quad p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1 \cdots n.$$

The transformed equations on T^*Q are

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + A(q)\,\lambda \\ 0 &= A^T(q)\,\frac{\partial H}{\partial q}(q, p). \end{aligned} \tag{16}$$

This paper provides a coordinate change that removes the dependence of the equations on λ. Choose a matrix whose columns span the null space of A^T and call it $S(q)$, *i.e.*,

$$A^T(q)\,S(q) = 0.$$

Such a matrix exists at least locally. Choose new coordinates,

$$\begin{aligned} \tilde{p}^1 &= S^T(q)\,p \\ \tilde{p}^2 &= A^T(q)\,\frac{\partial H}{\partial p} \end{aligned} \tag{17}$$

where $\tilde{p}^1 \in \mathbb{R}^{n-k}$ and $\tilde{p}^2 \in \mathbb{R}^k$. This is a coordinate transformation since it is assumed that

$$\det A^T(q)\frac{\partial^2 H}{\partial p^2}(q,p)A(q) \neq 0, \quad q \in Q. \tag{18}$$

Showing that it is a coordinate transformation comes down to showing that the matrix

$$\begin{pmatrix} S^T \\ A^T \frac{\partial^2 H}{\partial p^2} \end{pmatrix} \tag{19}$$

is full rank. A mechanical system is assumed in this exposition so that $\frac{\partial^2 H}{\partial p^2} = M^{-1}(q)$, the inverse of the positive definite mass matrix. In the original coordinates, the equations take the form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J\begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix} + \begin{pmatrix} 0 \\ A(q) \end{pmatrix}\lambda, \quad A^T(q)\frac{\partial H}{\partial q}(q,p) = 0, \tag{20}$$

where

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

is the standard matrix for the Poisson bracket. I used these equations to calculate the equations in the new coordinates. After a messy calculation, I confirmed that the equations are given by

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \\ \dot{\tilde{p}}^2 \end{bmatrix} &= \tilde{J}(q,\tilde{p})\begin{bmatrix} \frac{\partial \tilde{H}}{\partial q}(q,p) \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^1}(q,p) \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{A}(q) \end{bmatrix}\lambda, \\ \tilde{p}^2 &= 0 \end{aligned} \tag{21}$$

where $\tilde{H}$ is the Hamiltonian in the new coordinates. Through the same calculation, it is determined that $\tilde{A}$ is a full rank $k \times k$ matrix. The Hamiltonian in the new coordinates obtains a block structure where there are no cross terms between $\tilde{p}^1$ and $\tilde{p}^2$ so that $\frac{\partial \tilde{H}}{\partial \tilde{p}^2}$ at $\tilde{p}^2 = 0$ is 0. This calculation was performed for a mechanical system with an invertible mass matrix which is only a function of the coordinates. The equations on the constraint manifold.

$$M = \left\{(q,p) \in T^*Q | A^T(q)\frac{\partial H}{\partial p}(q,p) = 0\right\},$$

are then

$$\begin{pmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{pmatrix} = J_r\left(q,\tilde{p}^1\right)\begin{pmatrix} \frac{\partial H_r}{\partial q}(q,\tilde{p}^1) \\ \frac{\partial H_r}{\partial \tilde{p}^1}(q,\tilde{p}^1) \end{pmatrix} \tag{22}$$

where $(q,\tilde{p}^1)$ are coordinates for the constraint manifold, J_r is $\tilde{J}$ with $\tilde{p}^2=0$ and H_r is $\tilde{H}$ with $\tilde{p}^2=0$. The matrix J_r has the form

$$J_r\left(q,\tilde{p}^1\right)=\begin{pmatrix} 0_n & S(q) \\ -S^T(q) & \left(-p^T\left[S_i,S_j\right](q)\right)_{i,j=1,\cdots,n-k}\end{pmatrix} \tag{23}$$

where S_i is the ith column of S. This form of J_r comes out of the messy calculation mentioned earlier. The matrix J_r defines a bracket on M where

$$\{F_r,G_r\}_r\left(q,\tilde{p}^1\right)=\begin{pmatrix}\frac{\partial F_r^T}{\partial q}\frac{\partial F_r^T}{\partial \tilde{p}^1}\end{pmatrix} J_r\left(q,\tilde{p}^1\right)\begin{pmatrix}\frac{\partial G_r}{\partial q}\\ \frac{\partial G_r}{\partial \tilde{p}^1}\end{pmatrix}. \tag{24}$$

This bracket satisfies two properties of the Poisson bracket, namely, skew-symmetry and Leibniz' rule. The Jacobi identity is not necessarily satisfied. The major theorem of "On the Hamiltonian Formulation of Nonholonomic Mechanical Systems" is the following:

Theorem 3.1 *The bracket $\{,\}_r$ on M satisfies the Jacobi-identity if and only if the constraints $A^T(q)\dot{q}=0$ are holonomic.*

Proof. (If) Suppose the constraints are holonomic. It is then possible to choose local coordinates so that the constraints are give by $(\dot{q}_{n-k+1},\cdots,\dot{q}_n)=0$. In this form, $A^T(q)=\begin{bmatrix}0 & I_k\end{bmatrix}$. Since this is a point transformation, momentum coordinates $\tilde{p}=(\tilde{p}^1,\tilde{p}^2)$ can be chosen where $\tilde{p}^1$ is $n-k$ dimensional and $\tilde{p}^2$ is k dimensional so that the equations are given as in (20). With the choice of new coordinates given by (17), $J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$ transforms to

$$\tilde{J}=\begin{bmatrix} 0 & 0 & I_{n-k} & * \\ 0 & 0 & 0 & * \\ -I_{n-k} & 0 & 0 & * \\ * & * & * & * \end{bmatrix}$$

where $*$ elements are unspecified. This follows since $S^T(q)=\begin{bmatrix}I_{n-k} & 0\end{bmatrix}$. The J_r matrix is the upper left $2n-k$ block. Since J_r is constant, $\{,\}_r$ satisfies Jacobi's identity.

(Only If) Suppose the bracket $\{,\}_r$ satisfies the Jacobi identity. Denote the Hamiltonian vector fields with respect to J_r with Hamiltonians $q_1,\cdots,q_2,\tilde{p}^1,\cdots,\tilde{p}^{n-k}$ by $X_{q_1},\cdots,X_{\tilde{p}^{n-k}}$. The Hamiltonian vector fields are just the columns of J_r. The fact that

$$\left[X_{\tilde{p}^i},X_{\tilde{p}^j}\right]=-X_{\{\tilde{p}^i,\tilde{p}^j\}},i,j=1,\cdots,n-k$$

implies that $\left[X_{\tilde{p}^i},X_{\tilde{p}^j}\right]$ is in the span of the columns of J_r. This follows since $X_H=J(DH)^T$. The first n factors of the vector fields of the form $X_{\tilde{p}^i}$ are just the ith column of $S(q)$ denoted S_i. It is then seen that $[S_i,S_j]\in ImS(q)$ for $i,j=1,\cdots,n-k$. Therefore, the distribution $D(q)=ImS(q)$ is involutive and hence the constraints are holonomic by Frobenius' theorem.

□

In the next section, these calculations are demonstrated for the penny rolling vertically on a horizontal plane.

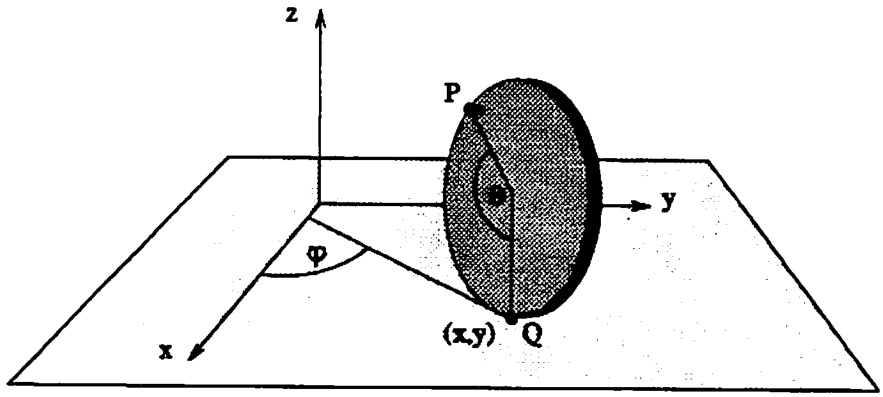

Figure 1: Penny Rolling Vertically on a Horizontal Plane

4 Penny Example

In this section. the penny rolling vertically on a horizontal plane is examined. The momentum equation is used to determine the equations of motion. The momentum equation leads to two conservation laws that along with the constraints completely solve the equations of motion. The calculations in Section 3 are carried out to determine the equations of motion on the constraint manifold and the bracket that does not satisfy the Jacobi equation. Next, the remainder in the Jacobi equation is calculated and related to the curvature of the connection determined by the constraints. The section ends by conjecturing a formula for the remainder in the Jacobi equation that involves the curvature of the connection. This penny system is simple but illustrates the ideas and leads to a conjecture for nonholonomic mechanical systems.

The notation for the penny model is now presented. The model for the vertical penny is shown in Figure 1. The coordinates of the point of contact, point Q, is denoted (x, y). The angle from Q to a reference point P measured about a normal on the flat face hidden from view and using the right hand rule is θ. The line tangent to the direction of the penny's heading intersects the x-axis. This angle is denoted φ. The fiber coordinates are chosen to be x and y and the base coordinates are chosen to be θ and φ. The mass of the penny is m, the moment of inertia about the θ axis is I, and the moment of inertia about the φ axis is J. The constraint equations are

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta}. \end{aligned} \tag{25}$$

In the notation of Ehresmann connections, $A_1^1 = -R(\cos\varphi)$, $A_1^2 = -R(\sin\varphi)$, and the remaining coefficients are zero. The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2. \tag{26}$$

4.1 The Penny and the Momentum Equation

First observe that the constraints (and the Lagrangian) are invariant under an $SE(2)$ action given by

$$(x, y, \theta, \varphi) \mapsto (x\cos\alpha - y\sin\alpha + d_x, x\sin\alpha + y\sin\alpha + d_y, \theta, \varphi + \alpha) \tag{27}$$

where (d_x, d_y, α) parameterize $SE(2)$. The $SE(2)$ action is a rotation about the z axis, the α variable, followed by a translation of the contact point, the d_x and d_y terms.

Also observe that the constraints (and the Lagrangian) are invariant under an $\mathbb{R}^2 \times S^1$ action given by

$$(x, y, \theta, \varphi) \mapsto (x + \lambda, y + \mu, \theta + \beta, \varphi\} \tag{28}$$

where (λ, μ, β) parameterizes the $\mathbb{R}^2 \times S^1$ group.

The tangent to the group orbits for the $SE(2)$ action is

$$T_q\text{Orb}_q^{SE(2)} = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \varphi}\right\}.$$

The tangent to the group orbits for the $\mathbb{R}^2 \times S^1$ action is

$$T_q\text{Orb}_q^{\mathbb{R}^2\times S^1} = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\right\}.$$

The distribution of allowable directions is given by the kernel of the constraint forms. As in the background presented in Section 2, the constraint distribution is denoted D_q. In this example,

$$D_q = \text{span}\left\{\frac{\partial}{\partial \varphi}, R\cos\varphi\frac{\partial}{\partial x} + R\sin\alpha\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}\right\}. \tag{29}$$

Calculating the intersections between the tangents to the group orbits and the constraint distribution gives

$$D_q \cap T_q\text{Orb}_q^{SE(2)} = \text{span}\left\{\frac{\partial}{\partial \varphi}\right\} \tag{30}$$

and

$$D_q \cap T_q\text{Orb}_q^{\mathbb{R}^2\times S^1} = \text{span}\left\{R\cos\varphi\frac{\partial}{\partial x} + R\sin\alpha\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}\right\}. \tag{31}$$

Now specify sections of the intersections described above. For the $SE(2)$, take the vector field to be

$$\xi_Q^q = \frac{\partial}{\partial \varphi} \tag{32}$$

with the corresponding Lie algebra element

$$\xi^q = (0, 0, 1)\,. \tag{33}$$

Notice that this Lie algebra element is independent of the fiber and the base coordinates and therefore leads directly to a conservation law. For the $\mathbb{R}^2 \times S^1$, take the vector field to be

$$\xi^q_Q = R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta} \tag{34}$$

with corresponding Lie algebra element

$$\xi^q = (R\cos\varphi, R\sin\varphi, 1)\,. \tag{35}$$

Notice that the third coordinate of the Lie algebra element is again independent of the base and fiber coordinates.

The momentum equation is now calculated for these two group actions. For the $SE(2)$ action, the nonholonomic momentum map is

$$J^c_\xi = \frac{\partial L}{\partial \dot{q}^i}\left(\xi^q_Q\right)^i = J\dot{\varphi}. \tag{36}$$

Since the right hand side of the momentum equation is zero, it follows that $\dot{\varphi} = \omega$ is a constant. For the $\mathbb{R}^2 \times S^1$, the momentum equation is

$$J^c_\xi = \frac{\partial L}{\partial \dot{q}^i}\left(\xi^q_Q\right)^i = m\dot{x}R\cos\varphi + m\dot{y}R\sin\varphi + I\dot{\theta}. \tag{37}$$

The momentum equation is then

$$\frac{d}{dt}\left(m\dot{x}R\cos\varphi + m\dot{y}R\sin\varphi + I\dot{\theta}\right) = m\dot{x}\frac{d}{dt}\left(R\cos\varphi\right) + m\dot{y}\frac{d}{dt}\left(R\sin\varphi\right). \tag{38}$$

Equation (38) reduces to

$$R\cos\varphi m\ddot{x} + R\sin\varphi m\ddot{y} + I\ddot{\theta} = 0. \tag{39}$$

Using the constraint equations to eliminate $\ddot{x}$ and $\ddot{y}$ gives

$$\left(mR^2 + I\right)\ddot{\theta} = 0. \tag{40}$$

This implies that $\dot{\theta} = \Omega$ is a constant. The equation for θ is $\theta = \Omega t + \theta_0$ and for φ is $\varphi = \omega t + \varphi_0$. Putting these into the constraints gives

$$\dot{x} = \Omega R\cos\left(\omega t + \phi_0\right)$$
$$\dot{y} = \Omega R\sin\left(\omega t + \phi_0\right).$$

Integrating these equations results in

$$x = \frac{\Omega}{\omega} R \sin(\omega t + \phi_0) + x_0$$
$$y = \frac{\Omega}{\omega} R \cos(\omega t + \phi_0) + y_0.$$

This example demonstrates the use of the momentum equation and shows that it agrees with other methods of generating the equations of motion. This example also provides one with intuition for the definitions presented earlier.

4.2 The "Hamiltonian" Equations for the Penny

For the following development, it is assumed for simplicity that $m = 1, I = 1, J = 1$, and $R = 1$. The Hamiltonian is $\frac{1}{2}\left(p_x^2 + p_y^2 + p_\theta^2 + p_\varphi^2\right)$. With this assumption, the constraints are $\dot{x} - \cos\varphi\dot{\theta} = 0$ and $\dot{y} - \sin\varphi\dot{\theta} = 0$. In this case,

$$A^T(q) = \begin{bmatrix} 1 & 0 & -\cos\varphi & 0 \\ 0 & 1 & -\sin\varphi & 0 \end{bmatrix},$$

and, therefore,

$$S(q) = \begin{bmatrix} 0 & \cos\varphi \\ 0 & \sin\varphi \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the coordinate change in equation (17) gives

$$\begin{aligned} \tilde{p}_1 &= p_\varphi \\ \tilde{p}_2 &= p_\theta + p_x\cos\varphi + p_y\sin\varphi \\ \tilde{p}_3 &= p_x - p_\theta\cos\varphi \\ \tilde{p}_4 &= p_y - p_\theta\sin\varphi. \end{aligned} \tag{41}$$

Furthermore, $M = \{(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi)\,|\tilde{p}_3 = 0, \tilde{p}_4 = 0\}$ and the bracket matrix is

$$J_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cos\varphi \\ 0 & 0 & 0 & 0 & 0 & \sin\varphi \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\cos\varphi & \sin\varphi & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

These equations can be inverted by hand or in Mathematica[7] to calculate the new Hamiltonian. It is seen that the new Hamiltonian has no cross terms between $\tilde{p}_1$ or $\tilde{p}_2$ and the $\tilde{p}_3$

and $\dot{p}_4$ terms. The new Hamiltonian on M is given by $H_r = \frac{1}{2}\tilde{p}_2^2 + \frac{1}{4}\tilde{p}_1^2$. The equations on M are

$$\begin{array}{rclcrcl} \dot{x} &=& \frac{1}{2}\tilde{p}_2\cos\varphi & & \dot{\tilde{p}}_1 &=& 0 \\ \dot{y} &=& \frac{1}{2}\tilde{p}_2\sin\varphi & & \dot{\tilde{p}}_2 &=& 0 \\ \dot{\theta} &=& \frac{1}{2}\tilde{p}_2 & & \dot{\tilde{p}}_3 &=& 0 \\ \dot{\varphi} &=& \tilde{p}_1 & & \dot{\tilde{p}}_4 &=& 0. \end{array}$$

4.3 The Remainder in Jacobi's Equation for the Penny

In this section. J_r in equation (42) is examined and the remainder in the equation in the Jacobi identity.

$$\{F,\{G,H\}_r\}_r + \{G,\{H,F\}_r\}_r + \{H,\{F,G\}_r\}_r, \tag{43}$$

is calculated.

To determine the remainder. only the remainder for the coordinate functions is calculated since the above equation is equivalent to

$$\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j}\frac{\partial H}{\partial x^k}\left[\left\{x^i,\left\{x^j,x^k\right\}_r\right\}_r + \left\{x^j,\left\{x^k,x^i\right\}_r\right\}_r \left\{x^k,\left\{x^i,x^j\right\}_r\right\}_r\right]. \tag{44}$$

To calculate the remainder, the following coefficients are determined:

$$C_{ijk} = \left\{x^i,\left\{x^j,x^k\right\}_r\right\}_r + \left\{x^j,\left\{x^k,x^i\right\}_r\right\}_r \left\{x^k,\left\{x^i,x^j\right\}_r\right\}_r. \tag{45}$$

Since there are three indices that range between 1 and 6, there is a potential of having $6^3 = 216$ coefficients. The cyclic nature in the coefficients leads to only 76 terms. Some of these 76 terms are related by minus signs. It is also useful to note that $\{x_i, x_j\}$ is the i,jth element of J_r. After performing this calculation and utilizing the relationships between the indices. only twelve nonzero terms remain. These twelve terms are

$$\begin{aligned} C_{165} = C_{651} = C_{516} &= \sin\varphi \\ C_{156} = C_{561} = C_{615} &= -\sin\varphi \\ C_{256} = C_{562} = C_{625} &= \cos\varphi \\ C_{265} = C_{652} = C_{526} &= -\cos\varphi. \end{aligned} \tag{46}$$

Using this result and placing it back into equation (44) gives

$$\begin{aligned} \text{equation}(43) &= \sin\varphi\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial \tilde{p}_2}\frac{\partial H}{\partial \tilde{p}_1} + \frac{\partial F}{\partial \tilde{p}_2}\frac{\partial G}{\partial \tilde{p}_1}\frac{\partial H}{\partial x} + \frac{\partial F}{\partial \tilde{p}_1}\frac{\partial G}{\partial x}\frac{\partial H}{\partial \tilde{p}_2}\right) \\ &- \sin\varphi\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial \tilde{p}_1}\frac{\partial H}{\partial \tilde{p}_2} + \frac{\partial F}{\partial \tilde{p}_1}\frac{\partial G}{\partial \tilde{p}_2}\frac{\partial H}{\partial x} + \frac{\partial F}{\partial \tilde{p}_2}\frac{\partial G}{\partial x}\frac{\partial H}{\partial \tilde{p}_1}\right) \\ &+ \cos\varphi\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial \tilde{p}_1}\frac{\partial H}{\partial \tilde{p}_2} + \frac{\partial F}{\partial \tilde{p}_1}\frac{\partial G}{\partial \tilde{p}_2}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial \tilde{p}_2}\frac{\partial G}{\partial y}\frac{\partial H}{\partial \tilde{p}_1}\right) \\ &- \cos\varphi\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial \tilde{p}_2}\frac{\partial H}{\partial \tilde{p}_1} + \frac{\partial F}{\partial \tilde{p}_2}\frac{\partial G}{\partial \tilde{p}_1}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial \tilde{p}_1}\frac{\partial G}{\partial y}\frac{\partial H}{\partial \tilde{p}_2}\right). \end{aligned} \tag{47}$$

After showing these equations to Professor Marsden, he conjectured that this equation might be equal to $\langle dF, B(X_G, X_H)\rangle$ + cyclic F, G, H where B is the Ehresmann curvature for the connection determined by the constraint forms and X_H is the Hamiltonian vector field for the pseudo-bracket. The connection is vertical valued and this means that for coordinate calculations,

$$\begin{aligned} \omega_1(X_G) &= [(dx - \cos\varphi d\theta)(X_G)]\frac{\partial}{\partial x} \\ \omega_2(X_G) &= [(dy - \sin\varphi d\theta)(X_G)]\frac{\partial}{\partial y}. \end{aligned} \tag{48}$$

These Hamiltonian vector fields are horizontal as can be checked or by seeing that the vector fields satisfy the constraints by construction. In local coordinates, the curvature of the Ehresmann connection is given by

$$B(X,Y) = dA(X,Y) - [A(X), A(Y)]. \tag{49}$$

Since the pseudo-Hamiltonian vector fields are horizontal, just take the exterior derivative of the constraints and apply them to the pseudo-Hamiltonian vector fields to calculate the curvature. Therefore,

$$\begin{aligned} B(X_G, X_H) &= d\omega_1(X_G, X_H)\frac{\partial}{\partial x} + d\omega_2(X_G, X_H)\frac{\partial}{\partial y} \\ &= (-\sin\varphi d\theta \wedge d\varphi(X_G, X_H))\frac{\partial}{\partial x} + (\cos\varphi d\theta \wedge d\varphi(X_G, X_H))\frac{\partial}{\partial y}. \end{aligned} \tag{50}$$

The matrix J_r is used to calculate the pseudo-Hamiltonian vector fields. I performed these calculations and found out that for this particular example,

$$\begin{aligned} &\{F, \{G, H\}_r\}_r + \{G, \{H, F\}_r\}_r + \{H, \{F, G\}_r\}_r = \\ &-\langle dF, B(X_G, X_H)\rangle - \langle dG, B(X_H, X_F)\rangle - \langle dH, B(X_F, X_G)\rangle. \end{aligned} \tag{51}$$

An equivalent way of writing this equation without a minus sign is

$$\begin{aligned} &\{\{G, H\}_r, F\}_r + \{\{H, F\}_r, G\}_r + \{\{F, G\}_r, H\}_r = \\ &\langle dF, B(X_G, X_H)\rangle + \langle dG, B(X_H, X_F)\rangle + \langle dH, B(X_F, X_G)\rangle. \end{aligned} \tag{52}$$

These calculations lead to the following conjecture for nonholonomic mechanical systems:

Conjecture 4.1 *Given a nonholonomic mechanical system with the constraints given by the horizontal distribution of an Ehresmann connection, the remainder after performing the calculations for the Jacobi identity is given by the following formula:*

$$\begin{aligned} &\{\{G, H\}, F\} + \{\{H, F\}, G\} + \{\{F, G\}, H\} = \\ &\langle dF, B(X_G, X_H)\rangle + \langle dG, B(X_H, X_F)\rangle + \langle dH, B(X_F, X_G)\rangle. \end{aligned}$$

where the bracket, $\{,\}$, *is the bracket on the constraint manifold.*

5 Conclusion

This report examined nonholonomic mechanical systems and was based on the three papers discussed in the introduction. This report demonstrated how tools in differential geometry are used to examine a nonholonomic system such as the vertical penny. The penny example in turn leads one to conjecture an equation for all nonholonomic systems. Further work involves understanding the role of connections in mechanics as well as control theory. Also, proving or disproving the conjecture is a step to be done. In conclusion, this report introduced the author to an interesting area and was a satisfying learning experience. It is hoped that this report will make it easier for anyone trying to understand any of the three papers.

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