Path Planning for a Satellite with Two Momentum Wheels

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December 13, 1990
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Abstract

This paper focuses on the problem of reorienting a satellite which has two momentum wheels. Viewed as a path planning problem, it bears some resemblance to parallel parking a car and these similarities will be explored. The path planning problem is manifestly non-holonomic. The kinematics for the satellite will be derived as will methods for finding inputs steering the system. Methods for choosing inputs minimizing the control effort which steer the system are also be presented. These are patterned loosely after the work of Montgomery[Mon90] on falling cats reorienting themselves as they fall.

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1 Introduction

Attitude control of a satellite can be achieved using carefully placed rocket thrusters. This method consumes rocket fuel and so has proven expensive. In addition, reservicing the craft often is out of the question, and so once the limited supply of rocket fuel is exhausted, control is lost. The solution to this problem is to attach rotors orthogonally to the craft. By spinning them, the attitude of the craft may be controlled.

We will be making several assumptions throughout the paper. The first two assumptions are mild in the sense that any satellite with rotors may be expected to satisfy them.

Assumption 1 The rotors are symmetric about their axis of rotation.

Assumption 2 The rotors are attached so that their axis of rotation will intersect and will do so at only one point. The one point is needed so we may conclude that the axes are not collinear.

Assumption 3 We will assume that total linear and angular momentum is zero with respect to the inertial frame we choose. These conditions may be dropped but it will complicate the discussion a great deal.

The paper itself is divided into two sections. The first derives the equations of motion for the satellite given our assumptions. The equations are written in a kinematic format to facilitate the path planning. The last section will give a steering algorithm and will explore both optimization and similarities to steering a unicycle.

2 The Kinematic Equations

We will represent the satellite with two rotors as three linked, rigid bodies. We will define the notation as it is needed. First, we will define the parameters describing physical system. Body 0 will be the satellite, the rotors being bodies 1 and 2 respectively. The orientation of each body with respect to a fixed inertial frame will be represented by the rotation matrix needed to rotate the body's frame onto the inertial frame. These will be denoted R_0, R_1, R_2 respectively. The space of such matrixes will be denoted $SO(3)$, or special orthogonal matrixes. Besides being orthogonal, these matrixes will be orientation preserving and thus have determinant equal to one.

Now we will specify the relative positions of each of the bodies. Draw a line through the axis of rotation of each rotor. Assume they intersect, and take that intersection to be the origin of the satellite body's frame. Set $d_0, d_1, d_2 \in \mathbb{R}^3$ to be the coordinates of the centers of mass of each body in this frame. Notice that for this system they are constants provided the rotors are symmetric. Set

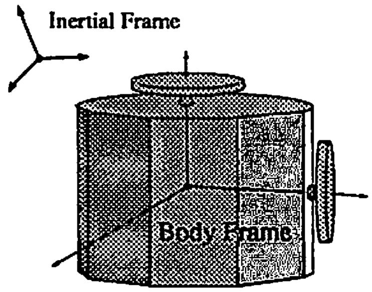

Figure 1: Satellite with Two Rotors

$c(t) \in \mathbb{R}^3$ to be the position of the origin of the satellite frame with respect to an inertial frame at some time t.

Any point in the satellite system may be specified in inertial frame coordinates by using the matrices R_0, R_1, R_2 and $c(t)$. So this will specify our configuration space Q. Define Q to be $SO(3)^3 \times \mathbb{R}^3$. Of course, the rotors themselves only have one degree of freedom so our choice of Q has four extra dimensions. We will worry about those hinge constraints later.

2.1 The Lagrangian

In the absence of any potential field, the kinetic energy is the Lagrangian. This can be computed by finding the mass and the square of the velocity of each point on the system with respect to the inertial frame, and then integrating in the usual way. Like Patrick [Pat90], denote $\rho_0(q), \rho_1(q), \rho_2(q)$ as the the density of each respective body at a point q as measured in the inertial frame when the system is in the *home* configuration. We may then write the Lagrangian as follows.

$$
\begin{aligned}
L &= \sum_{j=0}^{2} \frac{1}{2} \int_{\mathbb{R}^3} \left| \dot{R}_j q + \dot{R}_0 d_j + \dot{c} \right|^2 \rho_j(q) dq \\
&= \sum_{j=0}^{2} \frac{1}{2} \int_{\mathbb{R}^3} \left(\left| \dot{R}_j q \right|^2 + \left| \dot{R}_0 d_j \right|^2 + \dot{c}^2 \right) \rho_j(q) dq + \\
& \quad \sum_{j=0}^{2} \int_{\mathbb{R}^3} \left(\left(\dot{R}_0 d_j \right)^T \left(\dot{R}_j q \right) + \dot{c}^T \left(\dot{R}_j q \right) + \left(\dot{R}_0 d_j \right)^T \dot{c} \right) \rho_j(q) dq \\
&= \sum_{j=0}^{2} \frac{1}{2} trace \left(\dot{R}_j \int_{\mathbb{R}^3} q q^T \rho_j(q) dq \dot{R}_j{}^T + \dot{R}_0 d_j \int_{\mathbb{R}^3} \rho_j(q) dq d_j^T \dot{R}_0{}^T \right)
\end{aligned}
$$

$$+\frac{1}{2}\dot{c}^2 \int_{\mathbb{R}^3} \rho_j(q)dq + \left(\left(\dot{R}_0 d_j\right)^T + \dot{c}^T\right) \dot{R}_j \int_{\mathbb{R}^3} q\rho_j(q)dq$$
$$+ \left(\dot{R}_0 d_j\right)^T \dot{c} \int_{\mathbb{R}^3} \rho_j(q)dq$$

At this point we may substitute and eliminate all of the integrals. Define m_j and I_j to be the mass and inertial matrix respectively for the j^{th} body.

$$\begin{aligned} L &= \sum_{j=0}^{2} \frac{1}{2} trace\left(\dot{R}_j I_j \dot{R}_j{}^T + m_j \dot{R}_j d_j d_j^T \dot{R}_j{}^T + m_j \dot{R}_0 d_j d_j^T \dot{R}_0{}^T\right) + \frac{1}{2} m_j \dot{c}^2 \\ &\quad + m_j \left(\dot{R}_0 d_j\right)^T \left(\dot{R}_j d_j\right) + m_j \dot{c}^T \left(\dot{R}_j d_j\right) + m_j \left(\dot{R}_0 d_j\right)^T \dot{c} \end{aligned}$$

First thing to note is that $\dot{R}_j d_j$ will be zero for $j = 1, 2$ because of the choice of d_j. The vector d_j points along the axis of rotation of each rotor. Following the methods of Patrick [Pat90], we note that we can translate the inertial frame without changing the kinetic energy of the system. Thus $L(c, R_0, R_1, R_2, ...) = L(c + \delta c, R_0, R_1, R_2, ...)$ for all $\delta c \in \mathbb{R}^3$. This defines an action on the configuration space $l_{\delta c} : Q \to Q$ under which the value of the Lagrangian is preserved.

This tells us we can reduce the dimensions of the problem with the appropriate substitution. If we assume that linear momentum is zero, the problem will reduce as follows. Set CM to be the center of mass of the entire system. Let $m = m_0 + m_1 + m_2$ be the total mass of the system.

$$\begin{aligned} CM &= c + \frac{1}{m} R_0 \left(m_0 d_0 + m_1 d_1 + m_2 d_2\right) \\ \frac{d}{dt} CM &= 0 \\ 0 &= \dot{c} + \frac{1}{m} \dot{R}_0 \left(m_0 d_0 + m_1 d_1 + m_2 d_2\right) \\ \dot{c} &= -\frac{1}{m} \dot{R}_0 \left(m_0 d_0 + m_1 d_1 + m_2 d_2\right) \end{aligned}$$

At the end of this section we will collect all of the various assumptions and discuss them. The two facts simplify the expression for the Lagrangian after some algebra.

$$\begin{aligned} L &= \frac{1}{2} trace\left(\sum_{j=0}^{2} \dot{R}_j I_j \dot{R}_j{}^T + \dot{R}_0 \left(m_1 d_1 d_1^T + m_2 d_2 d_2^T + m_0 d_0 d_0^T\right) \dot{R}_0{}^T\right) \\ &\quad - trace\left(\dot{R}_0 \left(2\frac{m_0}{m} \left(m_0 d_0 d_0^T + m_1 d_0 d_1^T + m_2 d_0 d_2^T\right)\right) \dot{R}_0{}^T\right) \\ &\quad - trace\left(\dot{R}_0{}^T \left(\frac{m_1}{m} \left(m_0 d_0 d_1^T + m_1 d_1 d_1^T + m_2 d_1 d_2^T\right)\right) \dot{R}_0{}^T\right) \\ &\quad - trace\left(\dot{R}_0 \left(\frac{m_2}{m} \left(m_0 d_0 d_2^T + m_1 d_1 d_2^T + + m_2 d_2 d_2^T\right)\right) \dot{R}_0{}^T\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}trace\left(\frac{1}{m}\dot{R}_0\left(m_0^2 d_0 d_0^T + m_1^2 d_1 d_1^T + m_2^2 d_2 d_2^T\right)\dot{R}_0{}^T\right) \\
&+trace\left(\frac{1}{m}\dot{R}_0\left(m_0 m_1 d_0 d_1^T + m_0 m_2 d_0 d_2^T + m_1 m_2 d_1 d_2^T\right)\dot{R}_0{}^T\right)
\end{aligned}
$$

$$
\begin{aligned}
L \quad = \quad & \frac{1}{2m}trace\left(\dot{R}_0\left((m_0 m_1 + m_1 m_2) d_1 d_1^T + (m_0 m_2 + m_1 m_2) d_2 d_2^T\right)\dot{R}_0{}^T\right) \\
&+\frac{1}{2m}trace\left(\dot{R}_0\left((m_0^2 + 4m_0 m_1 + 4m_0 m_2) d_0 d_0^T - 4m_0 m_1 d_0 d_1^T\right)\dot{R}_0{}^T\right) \\
&-trace\left(\dot{R}_0\left(2m_0 m_2 d_0 d_2^T + m_1 m_2 d_1 d_2^T\right)\dot{R}_0{}^T\right) \\
&+\frac{1}{2}trace\left(\dot{R}_0 I_0 \dot{R}_0{}^T + \dot{R}_1 I_1 \dot{R}_1{}^T + \dot{R}_2 I_2 \dot{R}_2{}^T\right) \\
I_0' \quad := \quad & I_0 + \frac{1}{m}\left(\frac{1}{2}(m_0 m_1 + m_1 m_2) d_1 d_1^T + \frac{1}{2}(m_0 m_2 + m_1 m_2) d_2 d_2^T\right) \\
&+\frac{1}{m}\left(\frac{1}{2}\left(m_0^2 + 4m_0 m_1 + 4m_0 m_2\right) d_0 d_0^T\right) \\
&-\left(2m_0 m_1 d_0 d_1^T + 2m_0 m_2 d_0 d_2^T + m_1 m_2 d_1 d_2^T\right)
\end{aligned}
$$

I_0' is defined to incorporate all of the physical constants of the system involved with $\dot{R}_0$. Also, identify $TSO(3)$ with $SO(3) \times \mathbf{R}^3$ in the usual way. We are interested in expressing angular velocity in terms of body coordinates. In keeping with Patrick [Pat90], notice that in the following calculation that a similarity transform does not change the value of the eigenvalues, and hence does not change the trace of the matrix.

$$
\begin{aligned}
\frac{1}{2}trace\left(\dot{R}_j I_j \dot{R}_j{}^T\right) &= \frac{1}{2}trace\left(R_j^T \dot{R}_j I_j \dot{R}_j{}^T R_j\right) \\
&= \frac{1}{2}trace\left((\omega_j \times) I_j (\omega_j \times)^T\right) \\
&= \frac{1}{2}trace\left(\omega_j^T \omega_j I_j - \omega_j \omega_j^T I\right) \\
&= \frac{1}{2}\omega_j^T\left(trace(I_j) I - I_j\right)\omega_j \\
&:= \frac{1}{2}\omega_j^T J_j \omega_j \\
J_j &:= trace(I_j) I - I_j
\end{aligned}
$$

So in this manner define $J_0', J_1, J_2, \omega_0, \omega_1, \omega_2$. The final form of the Lagrangian, ignoring the hinge constraints, is given below. Notice that the Lagrangian in these coordinates is not dependent on the configuration.

$$
L \quad = \quad \frac{1}{2}\left(\omega_0^T, \omega_1^T, \omega_2^T\right)\begin{bmatrix} J_0' & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}\begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix}
$$

In deriving this formula, we have used many of our initial assumptions. If we do not assume that the axis of rotation for the momentum wheels intersect, the problem becomes more of a mess algebraically but is at heart the same problem. Dropping the assumption on zero linear momentum would only complicate the reconstruction of the state of the system but as before, the problem is not changed qualitatively. Dropping the assumptions that the rotors are symmetric about their axis of rotation, however, destroys the Lagrangian's independace of configuration. This changes the problem entirely.

2.2 The Momentum Map

Construct the following action on the configuration space by the group $SO(3)$. Given any $A \in SO(3)$ define the action to be $l_A : Q \to Q; (R_0, R_1, R_2) \to (AR_0, AR_1, AR_2)$. This action lifts to the velocity phase space of the system. Notice that the Lagrangian is invariant under this action, so we can form a momentum map to the lie algebra of the group giving us a new conserved quantity, namely the angular momentum which by Noether's theorem is conserved. Denote angular momentum by AM.

$$
\begin{aligned}
AM &= [III] \begin{bmatrix} J_0' & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix} \\
&= J_0' \omega_0 + J_1 \omega_1 + J_2 \omega_2
\end{aligned}
$$

Notice that J_1 and J_2 are the normal inertial matrixes, and that J_0' is merely and augmented version of the satellite's normal inertial matrix.

2.3 The Kinematic Equations

Now we will assume the total angular momentum is zero. This simplifies the dynamics greatly and makes reconstructing the state of the system from the reduced coordinates easy. Now we should apply the hinge constraints. In doing so, it is useful to define the normals in the direction of d_1 and d_2. The rate of each wheel will be denoted $\dot{\theta}_1, \dot{\theta}_2$ respectively.

$$
\begin{aligned}
n_1 &= \frac{1}{\| d_1 \|} d_1 \\
n_2 &= \frac{1}{\| d_2 \|} d_2 \\
\omega_1 &= R_1^T R_0 \left(\omega_0 + n_1 \dot{\theta}_1 \right) \\
\omega_2 &= R_2^T R_0 \left(\omega_0 + n_2 \dot{\theta}_2 \right)
\end{aligned}
$$

Given these hinge constraints, solve for the angular momentum in terms of $\omega_0, \dot{\theta}_1, \dot{\theta}_2$.

$$AM = J_0'\omega_0 + J_1 R_1^T R_0 \left(\omega_0 + n_1\dot{\theta}_1\right) + J_2 R_2^T R_0 \left(\omega_0 + n_2\dot{\theta}_2\right)$$

Notice that because of our choice of n_1 and n_2 and because the rotor is symetric about its axis of rotation, that the following identities will hold.

$$\begin{aligned} R_1^T R_0 n_1 \dot{\theta}_1 &= n_1\dot{\theta}_1 \\ R_2^T R_0 n_2 \dot{\theta}_2 &= n_2\dot{\theta}_2 \\ J_1 R_1^T R_0 &= J_1 \\ J_2 R_2^T R_0 &= J_2 \end{aligned}$$

Now we can write the angular momentum without the dependencies on configuration that we had earlier.

$$AM = \left(J_0' + J_1 + J_2\right)\omega_0 + J_1 n_1 \dot{\theta}_1 + J_2 n_2 \dot{\theta}_2$$

Define $V := \left(J_0' + J_1 + J_2\right)$ and assume it is invertible. In this way, we may solve for ω_0 and obtain the kinematic map.

$$\omega_0 = V^{-1} J_1 n_1 \dot{\theta}_1 + V^{-1} J_2 n_2 \dot{\theta}_2$$

If we consider the velocities of the rotors as our inputs to this system, we now have a kinematic model of the satellite. This is the model we will use to plan trajectories for the space craft. It is useful to define $b_1, b_2 \in so(3)$.

$$\begin{aligned} b_1 &= V^{-1} J_1 n_1 \\ b_2 &= V^{-1} J_2 n_2 \end{aligned} \tag{1}$$

The reconstruction of the actual state of the system is fairly straightforward with zero momentum. Given an initial configuration of the body, $A_0 \in SO(3)$, and the two inputs to the rotors $\dot{\theta}_1(t), \dot{\theta}_2(t)$ we can solve for the body twist $\omega_0(t)$ for any time.

The trajectory of the body 0's orientation will just be the solution of this differential equation.

$$\dot{R}_0 = R_0 \left(\omega_0 \times\right)$$

In the special case of piecewise constant $\omega_0(t)$; we can solve for the final orientaion more directly.

$$\begin{aligned} \xi &:= \omega_0(t) \forall t \in [0,1] \\ R_0(1) &= R_0(0) e^{(\xi \times)} \end{aligned} \tag{2}$$

This just coresponds to a twist in the body frame about the axis ξ by one radian. This solution will suffice for most of the motions the path planner will generate.

3 Parallel Parking the Satellite

This section develops an algorithm for steering the system from one orientation to another. We will assume that the positions of the rotors are not important, so the algorithm will only steer the orientation of the body. The algorithm developed will be in no way optimal. The last section considers the optimal case in the sense of control effort.

Already the kinematic equations developed in the last section look a great deal like the kinematic equations for the simple car model, the unicycle. The technique of steering by sinusoids of Murray and Sastry[RM90] may be applied here with success.

3.1 The Non-Holonomy

We will first establish the nonholonomic nature of the satellite and then prove controllability via Chow's Theorem[Isi89]. Suppose there existed some smooth function whose derivative was the two input vector fields of our control system.

Then, for the our manifold, that would imply that the cross product of the two input vector fields was zero at every point. However, this is not true for any point so it is falso that the input vector fields are the partials of some function.

Controlliblity is defined as folows. Given any initial and final configuration there exists some time T and some input defined on the interval $[0, T]$ so that the configuration at $t = 0$ is the starting one and the configuration at $t = T$ is the final one. For the case of the satellite, we only care about the final orientation of the satellite body, and so it will be considered controllable if any two orientoins can be joined by an admissable path through $SO(3)$.

For systems without drift like ours, Chow's theorem assures controllability if the involutive closure of the input vector fields spans the tangent space at every point in the configuration space.

Finding the involutive closure involves taking the Lie Brackets of the input vector fields. For our case only one level of Brackets are needed to span the tangent space at every point. Taking the Lie Bracket in $SO(3)$ amounts to taking the cross prodect of the twist axis. Provided that the are not dependent, this will trivially complete the basis.

Assumption number two of the introduction assures controllability by demanding that the axis of rotation intersect at one and only one point.

3.2 The Geometric Phase

The idea will be to reduce the configuration space to one in which we know how to move; one that is holonomic. Any configuration space of this type we will call shape space. Given that a system was controllable, the task will now focus on the job of finding out how movement in the shape space relates to motions in the configuration space.

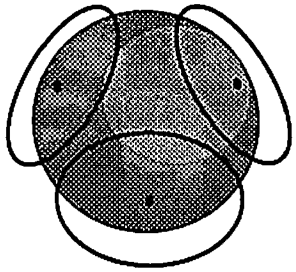

Figure 2: The Rotation Group as a Circle Bundle

The process of finding the unreduced state given a path in shape space and an intial condition is called reconstruction. The change that can not be seen in the reduced state will be denoted the geometric phase of the path.

From a path planning standpoint, the division of the space into the holonomic and nonholonomic parts clarifies the issue. In general, the reconstruction proccedure will provide the link and tell the planner how to move in shape space as to affect the desired changes in position in configuration space.

For the problem of the satellite, we will again ignore the orientation of the rotors. As before, the orientation of the satellite body will be given by $R_0 \in SO(3)$. Define b_0 as follows.

$$b_0 \quad = \quad \frac{b_1 \times b_2}{\| b_1 \times b_2 \|}$$

To form the shape space, identify any two configurations related to each other by any rotation through the axis given by b_0. Shape space is then $SO(3)/S^1 = S^2$. To visualize this space, one can think of Poncaire's representation os $SO(3)$ as a circle bundle over the two shere. Shape space may be embedded in $\mathbb{R}^3$ by the projection map $SO(3) \rightarrow S^2 \in \mathbb{R}^3 : R_0 b_0$ It may be easily checked that in this space the system may move holonomically, for $R_0 b_1, R_0 b_2$ span the tangent space by constuction.

This choice of the shape space makes the reconstruction straight forward. Suppose the path given is a positively oriented simple closed curve. Then the geometric phase for the path is merely the area enclosed by the curve on the shape space as embedded in $\mathbb{R}^3$.

Consider the projected initial point of the path, and attach a reference frame which will span the tangent space of the unit sphere at that point. As we travel along this path, this frame will keep track of any spin about the b_0 axis. But, by construction, the input vector field may induce spins about axes which are strictly perpendicular to the b_0 axis and thus there can be no spin about the b_0 axis. Therefore, the frame is parallel transported along this path. Assume the path is constructed out of finitely many smooth segments parameterized by arc length. Label each one of these segments C_i and the region they enclose R. An application of the Gauss-Bonnet Theorem [Car84] will then finish the proof, for it states:

$$2\pi\chi(R) = \sum \int_{C_i} k_g(s)ds + \int_R K d\sigma + \sum \theta_i$$

Where $\chi(R)$ is the Euler-Poincaré characteristic of the region, K being the curvature at each point, k_g being the geodesic curvature of curve C_i, and θ_i being the exterior angle at each discontinuity in the path.

In this case, $\chi(R)$ is equal to 1, so we may disregard the left term with $\chi(R)$ for it is a multiple of 2π. In addition, the curvature of the unit sphere is constant and equal to 1 therefore the surface integral will just give us area enclosed by the path. Finally the two summations give the net phase shift in the parallel transported frame. □

This result may be generalized to more complicated paths in the following manner. We will divide the problem into three increasingly complicated pieces.

Case 1 The path is a simple closed curve. We will also assume it is positively oriented. If not, the sign of the geometric phase is reversed. The rotation around the b_0 axis is then just the equal to the area enclosed by the path.

Case 2 The path is a self-intersecting closed curve. In this case the path may be broken into simple closed curves, some positively oriented and others negatively oriented. Apply case 1 to each piece, and add the result together to obtain the amount rotated around the b_0 axis.

Case 3 The path is an open smooth curve. Then, join the end points by a geodesic. In this case, this would just be a great circle. This will form the second case. From the result gained by appling case 2, subtract the phase generated by moving along the geodesic, which although is not zero is fairly simple to compute.

The path planner will only use the first case in little steps. For the linearized unicycle of Murray and Sastry[RM90], the shape space is the cylinder, $\mathbb{R} \times S^1$, and once again the geomtric phase they obtain by motions in shape space will be equal to the area enclosed by the the path, modulo the sign. Not every system has such a simple relationship. A more subtle system with a two dimensional shape space would be the system of three linked, rigid bodies skating on ice with motors at the joints. Here the shape space is the two torus, and the geometric phase is equal to a density function integrated over the area enclosed by the loops. The density factor for our case and for the linearized unicycle is just a constant.

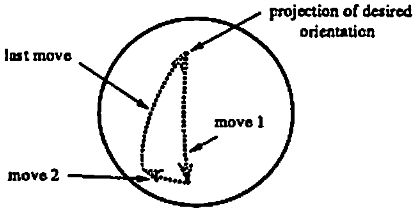

Figure 3: The Parallel Park Maneuver for the Satellite

3.3 Steering Algorithm

Given any starting configuration $A_0 \in SO(3)$ and any final configuration $A_1 \in SO(3)$, a two step planner can always find a path. Because the density factor of the geometric phase is always equal to a constant, sinusoidal steering will work starting in any configuration. The procedure is then as follows.

Step 1 Drive the two input fields withe constant inputs as to drive the system in shape space coordinates to the destination configuration mod the S^1 bundle coordinate.

This is akin to lining the unicycle up in front of the desired parking space before the parallel park manuver. In that case, the sidewise direction of the car is the difficult direction to move in and in this step is ignored.

Step 2 Now that we have lined the system up in shape space, all that is left to do is move along the fiber attached at this point. Keep a note of how far this error is. Now drive the system through the b_1 vector field until $R_0 b_0$ has moved through 90^{deg}.

Now move in along the great circle perpendicular to the last motion. Move the system until we have rotated through the same amount as the error. The final leg will just be driving along the geodesic connecting us back to $A_1 b_0$.

The second step is akin to parallel parking the linearzed unicycle. Because shape space there is a cylinder, four instead of three straight line pieces are required to enclose a non-trivial section of the space. This would coorespond to driving forward, then tturning the wheels, driving backwards, then returning the wheels to their original postion.

Of course one may commute the actions and get the same result. So you could turn the wheel, drive forward, trun the wheel back, and drive backwards. This encloses the same area in shape space and yeilds the same geometric phase.

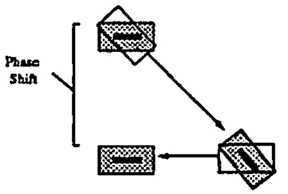

Figure 4: Parallel Parking for the Unicycle

3.4 Optimal Steering

Now that we know a path can be found, the next question is how to find the optimal path in regards to control effort. If friction is the bearings of our momentum wheels dominates the cost of moving the rotors, the cost is them given by:

$$Cost = \int_0^1 \left(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2}\right) dt$$

To solve this problem easily we will make a few assumptions. They simplify the optimization.

Assumption 4 The rotors are identicle except being mounted orthogonally.

Assumption 5 The axis of rotation of the rotors intersect at the center of mass of the satellite.

This algorithm will find the optimal loop in shape space given we need a some geometric phase. In this case, minimizing this integral is the same as finding the shortest length path in the shape space becuase of Assumption 5. With that in mind, the optimization becomes as easy as the the optimization of the loop for the linearized unicycle of Murray and Sastry [RM90]. For the unicycle, shape space was T^2 and the optimal trajectories were merely circles as a circle encloses the most area with the minimum perimeter.

For the satellite, the answer is the same. A circle on the two sphere which will enclose the maximum area with a minimum perimeter. The solid angle of the circle completley detrmines it. The geometric phase is given by a simple formula. Let ϕ be the solid angle.

$$phase = 2\pi\left(1 - \cos(\phi)\right)$$

This solution is akin to the optimization of Richard Montgomery[Mon90] on the falling cat.

References

[Car84] Manfredo P. Do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, second edition, 1984.

[Isi89] A. Isidori. *Nonlinear Control Systems.* Springer-Verlag, second edition, 1989.

[Mon90] R. Montgomery. Isoholonomic problems and some applications. *Communications in Mathematical Physics*, 128:565 –592, 1990.

[Pat90] G. W. Patrick. *Two Axially Symmetric Coupled Rigid Bodies: Relative Equilibria, Stability, Bifurcations, and a Momentum Preseving Sympletic Integrator.* PhD thesis, University of California at Berkeley, 1990.

[RM90] S. Sastry R. Murray. Grasping and manipulation using multifingered robot hands. Technical report, University of California at Berkeley, 1990.