1999

My original goal at the beginning of this endeavor was to examine nonequivariant momentum maps and how to define a central extension in order to make them equivariant. I also wanted to look at other aspects of nonequivariance. As the principle text I used Soriau [1970], unfortunately I did not receive it until Wednesday evening because it had been checked out of the library. In a moment of desperation I finagled the borrowers name from the librarian to get it. I have spent the remainder of the time attempting to understand the terminology of the book and where the equivariance came up due to the different language. Instead of providing the report I wanted to, I have here the translation into Marsden-ease of what I did get from the book. I feel a little guilty because this seems like a copout, but time restraints have kept me from putting in all I desired. Some of it may seem like a rehash, but it was learning for me. I suppose in the end that's all that counts.

I apologize for it being late, but the actual result that I was looking for did not have a large blinking sign and required a lot of reading between the lines. In fact it simply shows up as a comment in the book. I guess there's a lesson somewhere in that. ☺

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1 Introduction

Let us begin with the definitions as obtained from Marsden[1999].

We are given a Lie algebra $\mathfrak{g}$, which acts (canonically on the left) on a Poisson manifold P.

Definition 1.1. Given a linear map $J : \mathfrak{g} \to \mathcal{F}(P)$ that satisfies $X_{J(\xi)} = \xi_P$ for all $\xi \in \mathfrak{g}$, the momentum map $\mathbf{J} : P \to \mathfrak{g}$ of the action of the Lie algebra $\mathfrak{g}$ on the Poisson manifold P, is defined by

$$\langle \mathbf{J}(z), \xi \rangle = J(\xi)(z) \tag{1.1}$$

Definition 1.2. A mapping $\mathbf{J}: P \to \mathfrak{g}^*$ is **equivariant** if $\mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J} = \mathbf{J} \circ \Phi_g$ (1.2)

Closely related to the idea of equivariance is infinitesimal equivariance.

Definition 1.3. A mapping $\mathbf{J} : P \to \mathfrak{g}^*$ and its related mapping $\mathrm{J} : \mathfrak{g} \to \mathcal{F}(P)$ are called *infinitesimally equivariant* if one of the following equations holds:

$$J([\xi, \eta]) = \{J(\xi), J(\eta)\} \tag{1.3}$$

or

$$T_z \mathbf{J} \cdot \eta_P(z) = -ad^*_\eta \mathbf{J}(z) \tag{1.4}$$

Infinitesimal equivariance is related to the vanishing of the $C(P)$-valued 2-cocycle Σ. Equivariance implies infinitesimal equivariance by differentiating (1.2).

One thing to note about the case that Souriau treats is that it is on symplectic manifolds. A symplectic manifold is already imbued with a Poisson structure as determined by the symplectic form Ω.

$$\{F, G\} = \Omega(X_F, X_G) \tag{1.5}$$

For the momemtum map to be equivariant in the symplectic case means that it simply has to be a canonical transformation with respect to the symplectic form, ie.

$$\varphi^*\{F, G\} = \{\varphi^* F, \varphi^* G\} \tag{1.6}$$

When the cohomology class is not equal to zero, the momentum map fails to be symplectic. The goal is to then extend the symplectic manifold M to another one M' for which the cohomology is zero. How this ocurs in Souriau [1970] is not obvious.

Let's go to the treatise of this topic by Souriau. Prior to delving in, we need some background information since Souriau does not tackle the problem directly. Instead he chooses to define a prequantum manifold and derive the result from that. Before the prequantum manifold I go through some stuff about dynamical groups and cohomologies leaving out many (almost all) of the proofs due to time constraints, although some are pretty in their own right.

2 Dynamical Groups

Definition 2.1. A *presymplectic manifold U* is a manifold on which the differentiable 2-form Ω has $\ker(\Omega)$=constant dimension >0 and $\mathbf{d}\Omega = 0$.

Definition 2.2. Let (M,Ω) be a symplectic (or presymplectic manifold). A Lie group G acting canonically on M is a *dynamical group*. (A canonical action will mean from the symplectic perspective, eq (1.6))

Given the above definition, there is a natural way to obtain a moment of the G-action, otherwise called a momentum map.

Definition 2.3. If G is a dynamical group of a symplectic or presymplectic manifold (M,Ω), we will call μ the *moment* of the G-action if there exists a *momentum mapping* $\mathbf{J} : P \to \mathfrak{g}^*$ such that

$$\Omega^\flat(\xi_P) \equiv -\mathbf{d}[\mathbf{J}(z)\cdot\xi] \qquad \forall\xi\in\mathfrak{g} \tag{2.1}$$

and $\mu = \mathbf{J}(z)$. If M is symplectic then the moment can also be defined by vector field equation

$$\xi_P \equiv X_{\mathbf{J}\cdot\xi} \tag{2.2}$$

Sometimes we will use μ for $\mathbf{J}(z)$. This is the form that Souriau prefers.

Theorom 2.1. Let G be a dynamical group of a symplectic manifold (M,Ω).

a) If G possesses a moment μ, then adding a constant $\alpha \in \mathfrak{g}^*$ results again in a moment. All moments of G are obtained this way if M is connected.
b) If M is Hausdorff and simply connected, then every dynamical group possesses a moment.
c) If the Lie algebra of G coincides with its derived algebra, then G possesses a moment.

Proof,

a) If $\mathbf{J}$ and $\mathbf{H}$ satisfy (7), then $\mathbf{d}[(\mathbf{J}-\mathbf{H})(z)\cdot\xi]=0$, thus $(\mathbf{J}-\mathbf{H})\cdot\xi$ is constant and since M is connected we have that **J-H** is constant since ξ is arbitrary.

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Suppose that the symplectic form Ω arises from the exterior derivative of a 1-form Θ, the above definitions result in a momentum map definition using the 1-form.

Theorem 2.2. Let G act on a symplectic manifold (M,Ω), where the symplectic form is exact and is derived from $\Omega = -\mathbf{d}\Theta$ such that the action of G leaves Θ invariant. Then G is a dynamical group of M and possesses a moment μ defined by

$$\mu\cdot\xi = \mathbf{J}(z)\cdot\xi \equiv \Theta(\xi_P) \tag{2.3}$$

Proof.

That G acts canonically is given since exterior differentiation and push-forward are interchangeable. Take the Lie derivative of Θ with respect to ξ and apply Cartan's formula to obtain the result, since the Lie derivative is equal to zero.

■

Theorem 2.3. (Noether's). Let M be a prsymplectic manifold and let μ be a moment of a dynamical group of V. Then μ is constant on each leaf of the characteristic foliation of M.

3 Cohomology of Dynamical Groups

Recall that when G acts on a manifold, we define the action of G on M by, $\Phi_g : M \to M$.

The adjoint action of G on its Lie algebra $\mathfrak{g}$ is given by $Ad_g : \mathfrak{g}\to\mathfrak{g}$, $Ad_g(\xi) = T_e(R_{g^{-1}} \circ L_g)(\xi)$

Related to adjoint is the coadjoint, $Ad_g^* : \mathfrak{g}^*\to\mathfrak{g}^*$, which is derived from $\langle \alpha, Ad_g\xi\rangle = \langle Ad_g^*\alpha, \xi\rangle$ for $\alpha \in \mathfrak{g}^*$, $\xi\in\mathfrak{g}$. This gives the coadjoint action $\Phi_a^* = Ad_{a^{-1}}^*$.

If the manifold M is a Banach space, then we refer to a linear action Φ_g as a (linear) representation.

Theorem 3.1. Let (M,Ω) be a connected symplectic (or presymplectic) manifold a let G be a dynamical group of M possessing a moment μ, whose momentum map is denoted by $\mathbf{J} : M\to\mathfrak{g}^*$. Then the following hold:

a) $\exists$ differentiable map θ:G$\to\mathfrak{g}^*$ defined by

$$\theta(a) \equiv \mathbf{J} \circ \Phi_a(x) - \Phi_a^* \circ \mathbf{J}(x) \tag{3.1}$$

b) The map θ satisfies the condition

$$\theta(ab) = \theta(a) + \Phi_a^* \circ \theta(b) \tag{3.2}$$

c) The derivative $f = D(\theta)(e)$, where e is the identity element of G, is a 2-form on the Lie algebra $\mathfrak{g}$ of G which satisfies:

$$f(\xi,[\eta,\zeta]) + f(\zeta,[\xi,\eta]) + f(\eta,[\zeta,\xi]) = 0 \tag{3.3}$$

d) The following identities hold:

$$D(\mathbf{J})(x)(\xi_P(x)) = \mathbf{J}(x) \cdot ad(\xi) + f(\xi) \tag{3.4}$$

$$\Omega(\xi_P, \eta_P) = \mathbf{J}([\xi,\eta]) + f(\xi,\eta) \tag{3.5}$$

Proof.

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4 Cohomology of Lie Groups

Definition 4.1. Let G be a Lie group and Φ_g a linear representation of G acting on E. An *E-cocycle* of G is any differentiable map θ from G to E such that

$$\theta(ab) \equiv \theta(a) + \Phi_a \circ \theta(b) \tag{4.1}$$

If for x ∈ E we define

$$\Delta(x)(a) = \Phi_a(x) - x \tag{4.2}$$

then $\Delta(x)$ is an E-cocycle of G called the *coboundary* of x.

(Sorry for not giving Δ)

Proof.

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Two cocycles of $\mathfrak{g}$ are said to be *cohomologous* if their difference is a coboundary. Cohomology is an equivalence relation in the vector space of E-cocycles of $\mathfrak{g}$.

5 Cohomology of Lie Algebras

Theorom 5.1. Let G be a Lie group, e its identity element, and $\mathfrak{g}$ its Lie algebra. Furthermore, let Φ_a be a linear representation of G. Then:

a) $\{\xi,\eta\} = [\xi_P, \eta_P]$ (5.1)

b) If θ is an E-cocycle of G and if we define $f = D(\theta)(e)$, the f is a linear map from $\mathfrak{g}$ to E satisfying

$$D(\theta)(a)(\xi(a) = \xi_E(\theta(a)) + f(\xi)$$

$$\theta(\exp(\xi)(e)) = \left[\int_0^1 \exp(t\xi_E)dt\right](f(\xi)) \tag{5.2abc}$$

$$f([\xi,\eta]) = \eta_E(f(\xi)) - \xi_E(f(\eta))$$

c) If G is connected and if f is zero, then θ is zero.

Proof.

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6 Symplectic Manifolds and Lie Groups

Let θ be an E-cocycle of a Lie group G. If we define

$$\Psi_a(x) = \Phi_a(x) + \theta(a) \tag{6.1}$$

it follows that the map Ψ_a defines an action of G on E.

Let $\mathfrak{g}$ be the Lie algebra of G. For $\xi \in \mathfrak{g}$ the associated vector field of Ψ_a is given by

$$\alpha = \xi(\mu) + f(\xi) \qquad (6.2)$$

where $f = D(\theta)(e)$ is the cocycle of the Lie algebra associated with θ.

Definition 6.1 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}_*$ will be called a *symplectic cocycle* of G if

$$\theta(ab) = \theta(a) + \Phi_a^* \circ \theta(b) \qquad [\theta \text{ is a } \mathfrak{g}^*\text{-cocycle}] \qquad (6.3)$$

$$f = D(\theta)(e) \text{ is antisymmetric}$$

Example. The Galilean Moment (Souriau pg. 144)

Let's look at the Galilei group diffeomorphic to $SO(3)\times\mathbb{R}^7$ given by:

$$a = \begin{bmatrix} A & \mathbf{b} & \mathbf{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \quad A \in SO(3), \mathbf{b}, \mathbf{c} \in \mathbb{R}^3, e \in \mathbb{R}$$

It's Lie algebra is given by

$$\xi = \begin{bmatrix} \hat{\omega} & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \quad \omega \in \mathbb{R}^3, \beta, \gamma \in \mathbb{R}^3, \varepsilon \in \mathbb{R}$$

The moment μ is given by:

$$\mu(\xi) = \langle \mathbf{l}, \omega \rangle - \langle \mathbf{g}, \beta \rangle + \langle \mathbf{p}, \gamma \rangle + E\varepsilon, \qquad \mathbf{l}, \mathbf{g}, \mathbf{p} \in \mathbb{R}^3, E \in \mathbb{R}$$

and we have,

$$\Omega(\xi_G, \cdot) = -\mathbf{d}[\mu \cdot \xi]$$

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The moment can be adjusted to the center of mass decomposition.

Theorom 6.1. Let G be a dynamical group acting on a connected symplectic manifold M and possessing a moment μ to which is associated a momentum map **J**. Let θ be the corresponding cocycle. Then,

a) Let G be a Lie subgroup of G with Lie algebra $\mathfrak{g}$. Then G is a dynamical group of V possessing a moment $\tilde{\mu}$ induced from μ:

$$\underline{\tilde{\mu}(\xi) = \overline{\mu}(\xi)} \qquad , \xi \in \mathfrak{g}$$

b) Suppose initially that G is a normal subgroup of G. Denote $\underline{\mathbf{J}}$ the map from M to μ, and by $\tilde{\Phi}_g^*$ the representation of G dual to the representation induced on $\mathfrak{g}$ by the adjoint representation of G, and by θ' the map

$$\theta'(\mathrm{a}) = \theta(\mathrm{a})$$

then

$$\underline{\mathbf{J}}(\Phi_a(x)) - \Phi_a^*(\underline{\mathbf{J}}(x)) = \theta'(a)$$

Proof.

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Theorom 6.2. Let G be a dynamical group of a connected symplectic manifold M possessing a moment μ. Assume that G is connected and abelian and that the 2-form f is injective. Then

a) θ is an isomorphism between the Lie group G and the additive group $\mathfrak{g}^*$.

b) If we define $\Omega(\alpha,\beta) = \alpha f^{-1}(\beta)$, g^* becomes a symplectic vector space.
c) The map J induces a symplectomorphism between each G-orbit in M and g^*.
d) M is symplectomorphic either to g^* or to the direct product $g^* \times M_0$ where M_0 is the submanifold of M defined by $\mu = 0$ with the induced symplectic structure.

Proof.

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7 Prequantum Manifold

Definition 7.1. Consider a Hausdorff manifold Y with a differentiable 1-form Θ. If the following properties are satisfied

$$\dim(\ker(d\Theta)) = 1 \tag{7.1}$$

$$\dim(\ker(\Theta)\cap\ker(d\Theta)) = 0 \tag{7.2}$$

the covector field given by Θ is said to define a *contact structure on Y.*

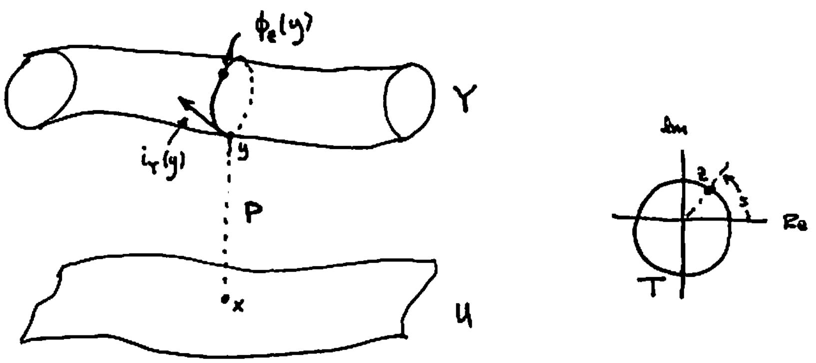
From Θ we can obtain a 2-form, $\Omega = -d\Theta$, whose rank must be even. Condition (7.1) then implies that the dimension of Y is odd. Since dime(ker(Ω)) is constant, Y is a presymplectic manifold, its leaves are curves (lines of force). Let us suppose that the curves are closed. It is possible to show that the action integral

$$a = \oint \Theta \frac{dy}{ds} ds \tag{7.3}$$

integrated over the leaf passing through y_0 satisfies $\partial a / \partial y_0 = 0$. Thus it is constant on every connected component of *Y*. We will say that *Y* is a prequantum manifold if $a=2\pi$. Let's introduce the vector $i_Y(y)$ defined by the two linear equations

$$\begin{aligned} &\Omega(i_Y(y)) = 0 \\ &\Theta(i_Y(y)) = 1 \end{aligned} \tag{7.4}$$

The first equation shows that $i_Y(y)$ is tangent to the line of force through y, the second gives its magnitude.



Definition 7.2. A Hausdorff manifold Y will be called a *prequantum manifold* if
a) There exists a differentiable covector field given by Θ on Y, which defines a contact structure on Y.
b) The torus T acts on Y in such a way that

$$\begin{aligned} &\phi_z : Y \to Y, \qquad \phi_z(y) = y \Leftrightarrow z = 1 \qquad\qquad z \in T \\ &\Omega(i_Y(y), \cdot) = 0 \\ &\Theta(i_Y(y)) = 1 \end{aligned} \tag{7.5}$$

where the action of z on Y is $\phi_z = \exp(s i_Y(\cdot))$, where $z = e^{is} \in T$. It should be noted that the structure of the prequantum manifold Y is entirely defined by the manifold structure of Y and the 1-form Θ since the action of T is defined using Θ and its exterior derivative Ω. Using the fact that the orbits of T are compact, one can show that the characteristic foliation of Y is sectionable. Therefore, the set U of lines of force admits the structure of a symplectic manifold whose symplectic form is defined by

$$\Omega(u,v) = \Omega(\xi,\eta) \tag{7.6}$$

where u,v are the projections of ξ,η of TY onto TU. U is called the base of the prequantum manifold Y, its dimension being one less than Y.

Conversely, given a Hausdorff symplectic manifold (U,Ω), we can construct a prequantum manifold Y. This process is called prequantization and results in a canonical transformation A between the base of Y and U. Denoting $x = A(\mathrm{Orb}(y))$ and $P: Y \to U$, $P(y)=x$, then

$$\begin{aligned}
&Y \text{ is a prequantum manifold}\\
&P \text{ is a differentiable map from } Y \text{ to } U\\
&\ker(D(P)(\xi)) \text{ is generated by } i_Y(y),\ \forall y \in Y\\
&x \in U \Rightarrow P^{-1}(x) \text{ is an orbit of T (acting on } Y)\\
&\Omega_Y(\xi,\eta) = \Omega_U(u,v)
\end{aligned} \tag{7.7}$$

On the other hand if we have a prequantum manifold Y and a map P satisfying (7.7) it follows that there exists a canonical transformation A from the base of Y onto U such that $P(y) = A(\mathrm{Orb}(y))$. Prequantizing a Hausdorff symplectic manifold U is the same as constructing a pair (Y,P) satisfying the axioms (7.7). When it is possible to make such a prequantization, then U is called prequantizable.

The following theoroms are given without proof.

Theorom 7.3. Let U be a Hausdorff symplectic manifold and let $\{U_i\}$ be an open cover of U such that (i) every U_j is simply connected and (ii) the symplectic form Ω admits a potential Θ on U_j. If U is prequantizable, there exist differentiable maps $x \mapsto z_{jk}$ from each of the (nonempty) $U_j \cap U_k$ to the torus such that

$$\Theta_k(u) - \Theta_j(u) = \frac{\delta z_{jk}}{i z_{jk}} \qquad \forall u \in T(U_j \cap U_k) \tag{7.8}$$

where δz_{jk} is a the tangent vector of z_{jk} and

$$\begin{aligned}
x \in U_j &\quad\Rightarrow\quad z_{jj} = 1\\
x \in U_j \cap U_k &\quad\Rightarrow\quad z_{jk} z_{kj} = 1\\
x \in U_j \cap U_k \cap U_l &\quad\Rightarrow\quad z_{jk} z_{kl} z_{lj} = 1
\end{aligned} \tag{7.9}$$

Theorom 7.4. Let U be a Hausdorff symplectic manifold and $\{U_i\}$ an open cover of U. Assume that there exist potentials Θ_j for the 2-form Ω defined on U_j, and differentiable maps $x \mapsto z_{jk}$ from $U_j \cap U_k$ to T satisfying (7.8) and (7.9). Then U is prequantizable.

If (U,Ω) is a symplectic manifold admitting a potential, $\Omega = -d\Theta$, then we can apply Theorom (18.22) with the open cover $\{U_i\}$ consisting of the single set U.

Theorom 7.5. Every Hausdorff symplectic manifold U admitting apotential is prequantizablr. Moreover, one can take the prequantum manifold Y to be the direct product $U \times T$ with

$$\begin{aligned}
&\Theta(u,\alpha) = \frac{\delta z}{iz} + \Theta_U(u)\\
&\phi_{\tilde z}(y,z) = (y, \tilde z z)\\
&P(y,z) = y
\end{aligned}$$

This result applies especially to the case when U is a symplectic vector space.

8 Quantization of dynamical groups

Let (Y,P) be a prequantization of the symplectic manifold U and let G be aLie group which acts canonically on U. It is clear that G is a dynamical group of the manifold U. It can be extended to the presymplectic manifold Y to become a dynamical group acting on Y via the equation

$$P(\Phi_g(y)) = \Phi_g(P(y)) \tag{8.1}$$

If the converse holds, G is a dynamical group for Y, then via the same equation (8.1) we can make G into a dynamical group for U. Using the theorom from the §2, we know that the prequantum manifold Y together with the dynamical group G possesses a moment. Since G must act on Y preserving the 1-form, using Theorom (2.2) it is possible to obtain a moment μ of G

$$\mu \cdot \xi = \Theta(\xi_Y(y)) \qquad \forall \xi \in \mathfrak{g}$$

and we know that the associated symplectic cocycle θ is zero. (11.21). We also note that via (2.2)

$$X_{\mu \cdot \xi}(y) = \xi_Y(y)$$

The generalized version of Noether's Theorom (Theorom 2.3) shows that μ depends on y only via x, and that μ is also a moment of G seen as a dynamical group of U. It is immediate that the cocycle associated with the latter moment is also zero, that is

$$\mathrm{J} \circ \Phi_a(x) = \Phi_a^* \circ \mathrm{J}(x)$$

We have shown the following proposition.

Proposition 8.1. If a dynamical group of a symplectic manifold is quantizable, then its symplectic cohomology is zero.

This final proposition gives us the conclusion we have long awaited.

Example. Let E be a symplectic vector space. As an additive group, E acts on itself by translations.

$$\Phi_a(x) = x + a \qquad \forall a, x \in E$$

from which we obtain,

$$\xi_E = \xi$$

The dynamical group E thus has the moment,

$$\mu = i_x \Omega$$

The associated cocycle is equal to Ω and thus its cohomology class is not zero. The symplectic structure of E therefore does not give an equivariant momentum map. E, however, admits a prequantiation (Y,P):

$$Y = E \times T$$

$$\Theta(u,\alpha) = \frac{\delta z}{iz} + \frac{1}{2} i_\xi \Omega(x)$$

$$P(x,z) = x$$

The dynamical group G still may act on Y, therefore using Proposition (**8.1**) we have that the prequantum manifold Y then has an equivariant momentum map, because it's symplectic cohomology class is zero. (How anticlimatic)

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