# Discrete Electromagnetism with DEC

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## 1 Introduction

My goal in this paper is to discuss problems in computational electromagnetism (E&M) as a motivation for more general theories of discrete differential geometry. In particular, the theory developed by Bossavit [1], [2], [3], has motivated—both historically and practically—the development of a discrete exterior calculus (DEC), [7], [6], with applications beyond just E&M.

First, I will discuss why we might wish to take a "geometric" approach to numerical problems such as these. Next, I will give a brief overview of Maxwell's equations, the centerpiece of E&M, and how we can get greater geometrical insight into them by using differential forms rather than ordinary vector calculus. I will then discuss Bossavit's treatment of computing solutions to Maxwell's equations (as presented in [1]) and show how this weak/finite element description has a natural correspondence to differential forms. After this, I will introduce the DEC operators that are pertinent to computational E&M. Finally, I will give a worked example of how to use DEC to derive discrete equations of motion from a variational principle for the spacetime Maxwell's equations.

## 2 Numerics that respect geometry

Why might we wish to take a geometric approach to numerics for physical systems? Why not simply pick an arbitrary method to discretize the equations of motion? In the continuous setting, the choice between a naive and a geometric point of view is often a matter of convenience and elegance; while a mechanical system may be easier to understand through, say, a Lagrangian formulation, one can (with some extra work) get the same results from $F = ma$.

This is not necessarily true, however, when we discretize. Many numerical methods have the property of "numerical dissipation," which, while it contributes to stability, destroys properties such as conservation of energy. We cannot hope for accurate numerics if our discrete system violates vital qualitative laws of the continuous system—such as planetary orbits spiraling inward! For this reason, we want to construct numerical methods carefully, so that the discrete equations of motion respect the geometric properties of the continuous system.

There are a number of properties we might have on a "wish list" for numerical methods, including the following:

1. The numerical algorithm is precisely the equations of motion for some discrete physical system, corresponding to the continuous problem we are computing.

2. If the continuous equations of motion preserve some variational principle, then the discrete equations of motion will preserve an analogous discrete variational principle.

3. Important geometric identities (e.g. Stokes' theorem) still hold after discretization.

4. The discretized system maintains the separation between metric-independent and -dependent components of the continuous system.

As we will see, both Bossavit's method and DEC address these issues favorably.

# 3 Maxwell's equations

Before we get into the problem of computing solutions to Maxwell's equations, let us first recall what the equations are [8], [1], [6]. Let $E$ be electric field intensity, $H$ be magnetic field intensity, $D$ be electric flux density, $B$ be magnetic flux density, $J$ be electric current density, and $\rho$ be electric charge density.

## 3.1 Vector calculus formulation

In ordinary vector calculus, these equations are often stated in either their integral form

$$\oint_P E \cdot dl = -\frac{d}{dt}\int_A B \cdot dA \tag{1}$$

$$\oint_P H \cdot dl = \frac{d}{dt}\int_A D \cdot dA + \int_A J \cdot dA \tag{2}$$

$$\oint_S D \cdot dS = \int_V \rho dv \tag{3}$$

where $A$ is a surface bounded by a path $P$ and $V$ is a volume bounded by a surface $S$, or in their differential form

$$\partial_t B + \operatorname{curl} E = 0 \tag{4}$$

$$-\partial_t D + \operatorname{curl} H = J \tag{5}$$

$$\operatorname{div} D = \rho. \tag{6}$$

We also relate $E$, $H$, $D$, and $B$ by the two constitutive equations

$$B = \mu H \tag{7}$$
$$D = \epsilon E. \tag{8}$$

Although the two statements of Maxwell's equations are equivalent, much of the geometric intuition is lost when we move from the integral to the differential formulation. $E$ and $H$ are quantities naturally integrated over paths; $B$, $D$, and $J$ over surfaces; and $\rho$ over volumes. Yet, the differential equations hide this geometric intuition by turning everything into vector fields and scalars. The constitutive equations are particularly puzzling from this viewpoint: How can path quantities become surface quantities through scalar multiplication?

### 3.2 Exterior calculus formulation

To restore this geometric intuition, we can regard the abovementioned "path," "surface," and "volume" quantities as (respectively) 1-, 2-, and 3-forms. Then Maxwell's equations take on the new form

$$\partial_t B + dE = 0 \tag{9}$$
$$-\partial_t D + dH = J \tag{10}$$
$$dD = \rho \tag{11}$$
$$B = \star_\mu H \tag{12}$$
$$D = \star_\epsilon E \tag{13}$$

This formulation is preferable for a number of reasons. Not only have we restored the geometric clarity of where these quantities "live," but the formerly disparate operations of div and curl are shown to be simply different versions of the exterior derivative $d$. Furthermore, we have resolved the confusion in the constitutive equations, as $\mu$ and $\epsilon$ are not really scalars but Hodge star operators; this also shows that the constitutive equations are intimately connected with our choice of metric.

## 4 Bossavit's approach

Bossavit's approach in [1] considers the vector calculus formulation of Maxwell's equations, discretized on an oriented simplicial complex. However, he does it in such a way that respects the geometrical structure of the system, and which is in fact equivalent to using differential forms. Since div, grad, and curl are not defined for discrete functions, Bossavit constructs a "weak" form of these operators by integrating them with respect to a test function $\phi$ and then integrating by parts. So far, this is the same as a standard finite element method.

The key distinguishing feature in this method is the choice of the so-called Whitney elements as the test functions $\phi$. Here, we define *separate* Whitney elements corresponding to the different-order simplices—vertices, edges, faces, and tetrahedra—in the discrete mesh. For a vertex $n$, we choose $w_n$ to be the usual "hat" function, which equals 1 at vertex $n$ and 0 at all other vertices. For an edge $e = \{m, n\}$, we use the vector field

$$w_e = w_m \nabla w_n - w_n \nabla w_m.$$

This has the property, similar to the hat function, that the cirulation of $w_e$ is 1 along edge $e$ and 0 over all other edges. This idea is extended to faces and tetrahedra, and the analogous hat function property holds for these as well.

There is a natural correspondence between this "weak" approach and the differential forms perspective: both differential forms and weak operators "want to be integrated" over regions of a particular dimension. By choosing, in effect, a basis for these differential forms, the Bossavit approach respects many of the geometric properties of the original system.

# 5 Discrete exterior calculus (DEC)

DEC defines the usual exterior calculus objects and operators on an oriented simplicial complex. Since these are then easily implemented as vectors and matrices, solving a differential equation—such as the differential forms statement of Maxwell's equations—is reduced to the problem of solving a linear system.

While we will only consider the operators which are relevant to E&M, the complete list (including $\sharp$, $\flat$, etc.) is developed by Hirani in [7]. A summary of the theory and implementation is also given in [6].

## 5.1 Differential forms

Discrete differential forms are defined as *cochains* on the simplicial complex, i.e. linear functions taking chains to $\mathbb{R}$. Considering only the basis elements for chains, we say that a $k$-form $\omega$ assigns each $k$-simplex $\sigma^k$ a number $\int_{\sigma^k} \omega$, also written $\langle \omega, \sigma^k \rangle$. We can then extend this to arbitrary chains of oriented $k$-simplices by linearity.

## 5.2 Exterior derivative

We can define the exterior derivative $d\omega$, integrated over a $(k+1)$-simplex, by integrating $\omega$ over the $k$-simplices on its boundary and summing. By construction, Stokes' theorem automatically holds.

## 5.3 Dual simplices

Each $k$-simplex $\sigma^k$ has an associated graph dual $*\sigma^k$, which is an $(n-k)$-simplex. When $n = 3$, for example, vertices are associated with dual tetrahedra, edges with dual faces, faces with dual edges, and tetrahedra with dual vertices. There are a number of ways to construct the dual complex; the circumcentric dual is particularly attractive, since primal and dual simplices are orthogonal to one another. Furthermore, we can define dual forms on this dual complex. Since this provides a natural pairing between $k$- and $(n-k)$-forms, we will see that this is an obvious way to define the Hodge star operator.

### 5.4 Primal-dual wedge

Given a $k$-form $\alpha$ and a dual $(n-k)$-form $\lambda$, we can define the wedge product $\alpha \wedge \lambda$ over the convex hull of $\sigma^k$ and $*\sigma^k$:

$$\begin{aligned} \langle \alpha \wedge \lambda, CH(\sigma^k, *\sigma^k) \rangle &= \frac{|CH(\sigma^k, *\sigma^k)|}{|\sigma^k||*\sigma^k|} \langle \alpha, \sigma^k \rangle \langle \lambda, *\sigma^k \rangle && (14) \\ &= \frac{1}{n} \langle \alpha, \sigma^k \rangle \langle \lambda, *\sigma^k \rangle && (15) \end{aligned}$$

where this last equality follows from the orthogonality property of the circumcentric dual [5].

### 5.5 Hodge star

Finally, we define the Hodge star as

$$\langle \star\alpha, *\sigma^k \rangle = \frac{|*\sigma^k|}{|\sigma^k|} \langle \alpha, \sigma^k \rangle.$$

The scaling factor preserves the important property

$$\alpha \wedge \star\beta = \langle\!\langle \alpha, \beta \rangle\!\rangle \mu^n$$

$\alpha$ and $\beta$ are both $k$-forms and $\mu^n$ is the volume form. This clearly demonstrates why the Hodge star is a metric-dependent operator.

### 5.6 Interpolation via Whitney forms

While we now have all the equipment we need to solve Maxwell's equations as a linear system, our solution will still be in terms of DEC-style discrete differential forms. If we wish to interpolate this solution back into the continuous setting, we can do so using Whitney forms—the differential forms analog of the Whitney elements defined above. (For example, the edge forms replace the $\nabla$ from the edge elements with $d$.) This combination of DEC operators with Whitney form interpolation precisely replicates Bossavit's method [6].

### 5.7 The advantages of DEC

The DEC approach has a number of advantages over other numerical methods. First, DEC provides a clear geometric picture of what forms are and where they "live"—arguably, clearer than smooth differential geometry, especially regarding the dual nature of the Hodge star. DEC also preserves many of the key theorems, identities, and metric-dependencies of smooth differential geometry. Additionally, the DEC operators and objects have a straightforward implementation as matrices and vectors, which immediately "translates" a differential equation into a linear system. Finally, the DEC approach is not limited to E&M; it generalizes to other problems and formulations.

# 6 Example: Maxwell's equations in spacetime

We can also use DEC to consider a 4-dimensional spacetime formulation of Maxwell's equations, and show that the resulting discrete equations correspond to a discrete variational principle. (This example is taken from [5].) Here, we look at the simplified case of a E&M in a vacuum with no sources.

## 6.1 Smooth formulation

With the Lorentzian metric on $X = M \times \mathbb{R}$, given the potential 1-form $A$, we have the action functional

$$\mathcal{S} : A \mapsto \frac{1}{2} \int_X \|dA\|^2 \, \mathbf{v} = \frac{1}{2} \int_X dA \wedge \star dA.$$

Taking its variation gives us (after applying Stokes' theorem and shuffling the Hodge stars)

$$\begin{aligned} \frac{1}{2} \int_X dA \wedge \star d\delta A + d\delta A \wedge \star dA &= \frac{1}{2} \int_X d \star dA \wedge \delta A - \delta A \wedge d \star dA && (16) \\ &= \int_X d \star dA \wedge \delta A && (17) \end{aligned}$$

so the equations of motion are $\star d \star dA = 0$. If we let $F = dA$, the field strength 2-form combining both $E$ and $B$, we get the well-known equivalent formulation

$$dF = 0, \quad \star d \star F = 0 \tag{18}$$

## 6.2 DEC formulation

We can replicate this calculation with DEC by defining a discrete action functional

$$\mathcal{S}_d : A \mapsto \frac{1}{2} \sum_{\sigma^2} \langle dA \wedge \star dA, CH(\sigma^2, *\sigma^2) \rangle.$$

To take variations, we look at the 1-form basis elements $\eta$, which assign 1 to a single edge $\sigma_0^1$ and 0 to all others. Then letting $A_\epsilon = A + \epsilon\eta$, Hamilton's principle becomes

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{S}_d(A_\epsilon) = 0.$$

Now, define the Hodge dual to be

$$\langle *\alpha, *\sigma^k \rangle = \kappa(\sigma^k) \frac{| * \sigma^k |}{|\sigma^k|} \langle \alpha, \sigma^k \rangle$$

where $\kappa(\sigma^k)$ is the "causality sign," which equals 1 if all the edges of $\sigma^k$ are spacelike and $-1$ otherwise.

Because $A_\epsilon$ only varies over the edge $\sigma_0^1$, all other $\sigma^2$ terms not containing it drop, and this equation becomes

$$\langle \star d \star dA, \sigma_0^1 \rangle = 0, \text{ i.e. } \star d \star dA = 0$$

which is precisely the discretization of the smooth equations of motion [5].

# 7 Final thoughts and future work

Although the two methods presented here were developed in two separate frameworks—finite elements and differential geometry, respectively—they both share the same essential geometric concepts and features that we wish to have in a properly "geometric" numerical scheme. There is a stronger link, both historically and theoretically, between discrete differential forms and classic finite element numerics than one might initially suspect.

An interesting topic for future work would be the equivalence between the continuous and discrete equations of motion presented in the previous section. While the discrete and continuous equations of motion *may* be identical, as in this case, this is not true generally. For example, the Chern-Simons field, which has action functional $\mathcal{S} : A \mapsto \int_M A \wedge dA$, results in discrete equations of motion that are weaker than the continuous case [5]. It would be interesting to see what causes this (the use of the primal-primal vs. primal-dual wedge product?), and what the impact is on numerics.

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