CDS205 Final Project: An Attempt at a Unified Variational Principle for Dynamics and Optimal Control of an Underactuated Mechanical System

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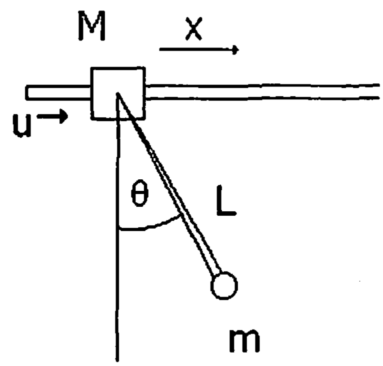
Abstract

I study the dynamics and optimal control of a simple moving-pendulum system which is actuated only at the base. Using two simple substitutions, I recast the optimization of the control variable as a higher-order Hamilton-like variational principle for the dynamical variables. The resulting Euler-Lagrange equations have an intuitively appealing structure, being fourth order in the actuated variable, but only second order in the unactuated variable. Furthermore, they demonstrate the translational symmetry present in the uncontrolled system, despite the fact that this symmetry is seemingly "broken" by the presence of control in the standard Hamilton's principle. This symmetry gives rise to a "higher-order" conserved quantity involving accelerations and jerks in addition to velocities and positions. Furthermore, the derivation of the conserved quantity gives a formula for an optimal feedback control law. The results give an overdetermined system of four differential equations for three functions, and I am unable to prove that they are compatible. As a surrogate for this analysis, I present an explicit solution for a linearized problem, and demonstrate that in this case, the equations are compatible.

Introduction: Pendulum on a Rail

The system of interest is a planar pendulum mounted on a brace which is free to translate along a fixed rail without friction. The rod is assumed massless. The brace and the bob have masses M and m respectively, and the length of the rod is L. Further, the brace is subject to a control force u, to be optimized in the L_2 norm. The system is illustrated below:



This system has two degrees of freedom, but is actuated only in the x variable. In particular, the virtual work is merely $\delta W = u dx$. The Lagrangian is as follows:

$$L(x,\dot{x},\theta,\dot{\theta}) = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}(L^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}\cos(\theta)) + mgL\cos(\theta)$$

Note that the Lagrangian is invariant under translations in x, suggesting a related conserved quantity. Unfortunately, the non-conservative contribution from the control u will prevent us from deriving a conserved quantity, but this situation will be rectified later. For now, we examine the Euler-Lagrange equations for Hamilton's principle, $\int_{t_1}^{t_2} L dt = \delta W$, which are as follows:

$$(M+m)\ddot{x} + mL(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)) = u$$

$$mL^2\ddot{\theta} + mL(\ddot{x}\cos(\theta) - \dot{x}\dot{\theta}\sin\theta) + mgL\sin\theta = 0$$

These are two second-order differential equations in three variables, $x(t), y(t), u(t)$. We observe that we can explcitly solve for $\ddot{\theta}$, and substitute it in the first equation. This gives:

$$(M+m)\ddot{x}+mL(\frac{1}{L}(\ddot{x}\cos(\theta)^2 - \dot{x}\dot{\theta}\sin\theta\cos\theta) - \frac{g}{L}\sin\theta\cos\theta - \dot{\theta}^2\sin(\theta)) = u$$

This substitution achieves two convenient goals. First, it embeds the dynamics of the unactuated variable in the equation for the actuated variable, hence including the coupling of control to the unactuated variable in a single equation. Second, it eliminates second derivatives of the unactuated variable, which proves convenient in the following section.

Optimal Control

We are interested in optimizing the control in the L_2 sense. In particular, the cost functional we minimize is simply $J(u) = \int_0^T u^2 dt$. Note that this is a rather artificial cost functional, in the absence of any constraints, since the trivial zero control clearly optimizes it globally. Nonetheless, we will show that it admits a family of local minima. The result to follow is suggestive of possibilities in more serious problems.

The common wisdom for solving such a problem would treat the differential equations arising from Hamilton's principle as constraints on this optimization. In contrast, we seek to embed this information directly into the variational principle, by substituting our previously derived expression for u. This substitution allows us to view the cost functional as defined on the space of paths rather than on the space of admissible controls. The optimization now has the form:

$$\min_{x,\theta} \int_0^T [(M+m)\ddot{x}+mL(\frac{1}{L}(\ddot{x}\cos^2\theta-\dot{x}\dot{\theta}\sin\theta\cos\theta)-\frac{g}{L}\sin\theta\cos\theta-\dot{\theta}^2\sin\theta)]^2 dt$$

For notational convenience, we write the above as $\int F(\dot{x},\ddot{x},\theta,\dot{\theta})dt$.

We gloss over existence and uniqueness problems for optima. Although the original optimization problem, appended to convex constraints, is a convex optimization, the modified optimization need not be convex in the new variables x, θ. We comment that if u is determined by an affine function of x and θ, then the convexity of the optimization is preserved, but this is not a necessary condition. However, this proves to be exactly the case in the linearized analysis to follow.

Now, a simple directional-derivative argument can derive the following necessary condition for local optimality of a candidate pair x, θ in an unconstrained setting (the presence of constraints would merely add the usual Lagrange multiplier terms on the right-hand side):

$$\ddot{F}_{\ddot{x}} - \dot{F}_{\dot{x}} + F_x = 0$$

$$\dot{F}_{\dot{\theta}} - F_\theta = 0$$

These equations have the interesting property of being of mixed order. The actuated variable is governed by a fourth-order differential equation, whereas the unactuated variable is governed by a second-order differential equation. We observe further that the cost functional is independent of x, and hence the first equation reduces to:

$$\frac{d}{dt}(\dot{F}_{\ddot{x}} - F_{\dot{x}}) = 0$$

This very nicely satisfies our original expectation of a conserved quantity. Despite the fact that the system symmetry is broken by the presence of an arbitrary control function, we find that the optimal control reasserts the symmetry of the uncontrolled system. This satisfies intution in that it seems impossible to have an optimal control law which is not translation-invariant, given a translation-invariant plant.

One can explicitly calculate the conserved quantity to be the following:

$$u[2m\dot{\theta}\cos\theta\sin\theta] + 2\dot{u}[(M+m) - m\cos^2\theta] = K$$

Here u has been substituted again, both for the sake of brevity, and for the purpose of the following observation: derivation of a conserved quantity seems to have led us to a differential equation for an optimal control law. Rearranging the above gives:

$$\dot{u} = \frac{K - u[2m\dot{\theta}\cos\theta\sin\theta]}{[(M+m) - m\cos^2\theta]}$$

Some comments are in order at this point. This is a strictly causal feedback law, and is invariant under translations in x, as expected. While the latter is predicted by the above calculations, the former is in no way intuitive. Indeed, many problems in the calculus of variations give rise to solutions which are either open-loop or non-causal (e.g. having a control function depending explicitly on higher derivatives of the state). The origin of our fortuitous discovery is unclear. My only guess at this point is that causality is a consequence of the control law being derived from a conservation principle (which, intuitively, must be a "causal" principle).

The control law is clearly nonlinear, although one might be tempted to say that it follows the general philosophy of integral control, in that it appends a new dynamical variable to the system dynamics.

Now, we turn our attention to the second necessary optimality condition. This takes the form:

$$\begin{aligned} 2\dot{u}[-2mL\ddot{\theta} - m\ddot{x}\cos\theta\sin\theta + mg(\cos^2\theta - \sin^2\theta) + mL\dot{\theta}^2\cos\theta] \\ +2[m\dot{x}\sin\theta\cos\theta - 2mL\dot{\theta}]\dot{u} &= 0 \end{aligned}$$

Note that under our previous substitution for $\ddot{\theta}$, this equation is only second-order in θ as expected.

We now have four differential equations to be satisfied by three functions, and I have been unable to prove that these equations are consistent. The last equation is clearly *not* simply the Euler-Lagrange equation for θ, although the following simplified problem suggests that this second equation may actually be redundant.

Simplified Linearized Problem

Here I simply consider the pendulum linearized about the rest position. The Lagrangian is now:

$$L(x,\dot{x},\theta,\dot{\theta}) = \frac{M+m}{2}\dot{x}^2 + \frac{mL^2}{2}\dot{\theta}^2 + mL\dot{\theta}\dot{x} - \frac{mgl}{2}\theta^2$$

The governing equations are:

$$(M+m)\ddot{x} + mL\ddot{\theta} = u$$

$$mL^2\ddot{\theta} + mL\ddot{x} + mgL\theta = 0$$

Performing the same substitutions as above, we arrive at the following cost functional:

$$\int_0^T (M\ddot{x} - mg\theta)^2 dt$$

There is once again a conserved quantity resulting from optimality with respect to x, which is found to be:

$$\frac{d}{dt}2M(M\ddot{x} - mg\theta) = K$$

Note the presence of the time derivative. Here we can explicitly see that the conserved quantity will involve the jerk of x and the velocity of θ. In particular, the conserved quantity in this case is precisely equal to the derivative of the control implying that the optimal control is a linear function of time.

Now, the second necessary condition turns out to be:

$$2mg(M\ddot{x} - mg\theta) = 0$$

Substituting this result in the previous condition, we find that the conserved quantity is precisely zero. Furthermore, this can now be substituted into the original dynamics to obtain:

$$(M + m)\ddot{x} + mL\ddot{\theta} = 0$$

$$mL^2\ddot{\theta} + mL\ddot{x} + mgL\theta = 0$$

which is precisely the uncontrolled system. This is a trivial sanity check, since we already commented that zero control would globally optimize the cost functional. Note that the two optimality conditions were consistent with the governing dynamics. Further, the linearized problem admitted a unique local minimum. The significance of these two facts with regard to the nonlinear problem is unclear, but the former suggests that one might be able to eke out consistency with some clever calculations.

Conclusions

A simple underactuated mechanical system was studied. We have formulated a problem in the calculus of variations which putatively embeds Hamilton's variational principle in the broader scheme of optimal control, without having to treat the dynamics as a constraint. This led to a higher-order variational principle and corresponding mixed-order governing differential equations for the dynamical variables. The translational symmetry of the problem was observed as a conserved quantity in the optimal control problem, and this conserved quantity led directly to a causal feedback control law. The final result was a set of four equations for three dynamical variables of unproved consistency. Linear analysis suggested that one of the optimality constraints may in fact be redundant, but it is unclear how one might prove this.

References

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