CDS 205

Project

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This work is a brief

review of the literature.

It is NOT an original work.

Abstract:

When some one with a background in Solid Mechanics is studying Geometric Mechanics, it seems that, the two topics are from different planets! This work is to help making a bridge, and understanding that they are in a same planet, and in fact they are very close. They are at most on different sides of a river.

In this work, it is tried to restate some solid mechanics concepts via geometrical words.

May be one asks, why we should care about this bridge. One of the reasons is that, usually, it seems difficult to work with planes other than IR^3, e.g. curved shapes in solid mechanics, while using geometrical mechanics, one can use the concept of manifolds and write coordinates free formulas, which are not limited to IR^3, and can be used for arbitrary coordinate systems. This will have advantages mainly in numerical solid mechanics, where we are dealing with complex shapes and large time/dimension scales.

Definitions:

1-Reference configuration

Closure of an open set in IR^3, with piece wise smooth boundary.

2-Deformation (configuration)

A mapping $\Phi : B \to IR^3$ that is sufficiently smooth, orientation preserving, and invertible.

3- Cauchy stress vector

Force per unit *deformed* area at position *x* at time *t* across a surface element with unit normal ***n***.

4- First Piola-Kirchhoff stress vector

Force per unit *undeformed* area at position *x* at time *t* across a surface element with unit normal ***n*,** which is parallel to the Cauchy stress **t** (**x**, t, **n**).

5-Elastic materials:

Elastic material is a material, for which the first Piola-Kirchhoff stress can be written as $P(X,t) = \hat{P}(X, F(X,t))$ where ***F*** is the deformation gradient tensor, $F^i{}_j = \partial\Phi^i / \partial X^j$.

6- Hyperelastic materials:

If for an elastic material there exists a stored energy function $\hat{W}$ depending on points $X \in B$ and F, such that $\hat{\mathbf{P}} = \rho_{\mathrm{Ref}}\ \partial\hat{W} / \partial\mathbf{F}$, then the material is hyperelastic.

7-Elastostatics

Elastostatics is the elasticity at static situation, in which we only look at the end points of a deformation. The basic mathematical problem in elastostatics is to find a configuration $\Phi : B \to IR^3$ such that $DIV(P) + \rho_{\mathrm{Ref}}\, B = 0$ in the body B, and Φ is prescribed to be Φ_d on a portion ∂_d of $,\partial B$ and the traction $P.N = \tau$ is prescribed on the reminder ∂_τ .

8-Elastodynamics

In elastodynamics we consider time dependence of the motion, so the problem, which is a dynamic problem is finding a motion $\Phi(X,t)$ that satisfies Cauchy's equation of motion with prescribed boundary conditions and prescribed initial deformation $\Phi(X,0)$ and velocity $V(X,0)$. We can use Lagrange's variational or Hamiltonian equations in this case easily, in fact nonlinear elastodynamics is a Hamiltonian system.

Geometry in Elasticity

They are some major reasons that make it worthy to use geometry in elasticity. Some of them are:

1- Rod and shell theory. The work done in bending a flexible rod is proportional to the integral of the square of the curvature along the rod. The same is for shells which are 2-d objects, for more details see Antman [1972a] and Naghdi [1972].

2- One of the main concepts in elastodynamics and Continuum Mechanics is the rate of deformation tensor and its computation. This is equivalent to the important concept of the Lie derivatives in geometry.

3- Using the concepts of covariance and variance of geometry, we can get coordinate free equations, which are very useful in elasticity, mainly in computational works.

4- The other reason is the Variational or Hamiltonian structure of elasticity. In fact elasticity has this structure attached to it. Some of the benefits of variational structure of the elasticity are:

- It is easier in numerical computation.
- The equations in weak form hold in situations where the localized form doesn't make sense e.g. shock wave.
- It is mathematically helpful in the study of existence and uniqueness of solutions.

Linear Hamiltonian systems and classical elasticity

Let χ be a Banach space. We will have the following definitions.

a) A weak symplectic form on χ is a continuous bilinear map $\omega : \chi \times \chi \to IR$ that is skew and weakly nondegenerate. We speak of χ with a symplectic structure ω as a phase space.

b) A linear operator $A : D(A) \to \chi$ with domain $D(A)$ a linear subspace of χ is called Hamiltonian if it is ω skew.

c) The Hamiltonian or energy function of A is defined by $H(u) = \frac{1}{2}\omega(Au,u)$, $u \in D(A)$.

d) A bounded linear operator $T : \chi \to \chi$ is called canonical transformation if it preserves ω; that is $\omega(Tu,Tv) = \omega(u,v)$ for all v.

e) The Poisson bracket of H_A and H_B for each $x \in D(A) \cap D(B)$ is defined by: $\{H_A, H_B\}(x) = \omega(Ax, Bx)$

f) A symplectic manifold is a pair (P,ω) where P is a manifold modeled on a Banach space χ.

Lagrangian field theory and nonlinear elasticity

A Lagrangian density is a smooth map $\mathcal{L} : \Xi \to IR$ where

$\mathcal{L} = \mathcal{L}(X,\phi,\dot{\phi},F)$.

In the case of elasticity:

$$\mathcal{L}(\phi,\dot{\phi},D\phi) = \frac{1}{2}\rho_{\text{Ref}}\left\|\dot{\phi}\right\|^2 - \rho_{\text{Ref}} W(D\phi) - \rho_{\text{Ref}} \mathcal{V}_B$$

Define the Lagrangian $L : TQ \to IR$ associated to $\mathcal{L}$ and a volume element *dV(X)* on *B* by

$$L(\phi,\dot{\phi}) = \int_B \mathcal{L}(X,\phi(X),\dot{\phi}(X),F(X))\, dV(X) - \int_{\partial\tau} \mathcal{V}_\tau(\phi)\, d A$$

From Euler-Lagrange's equation, we get Lagrange density equation:

$$\frac{\partial}{\partial t}\partial_{\phi}\mathcal{L} = \partial_{\phi}\mathcal{L} - \mathrm{DIV}\,\partial_{D\phi}\mathcal{L} \qquad \text{in } B,$$

and the boundary conditions

$P.N = \tau$ on ∂_{τ}. Where

$P = -\partial\mathcal{L}/\partial D\phi$.

Generating function in elasticity

Let (P,ω) be a symplectic manifold, $\mathcal{L}$ a Lagrangian submanifold, and $i:\mathcal{L}\to P$ the inclusion. If, locally $\omega = -\mathrm{d}\theta$ then $i^*\omega = -\mathrm{d}i^*\theta = 0$ so $i^*\theta = \mathrm{dS}$ for a function $\mathrm{S}:\mathcal{L}\to IR$ (locally defined). We call S a generating function or a potential function for $\mathcal{L}$.

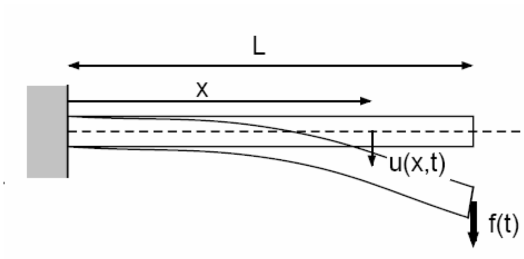
$\mathcal{L}$ is a Lagrangian submanifold of T*Q if and only if the stress is derived from an internal energy function W. The function $S(\phi) = \int_B W(F)\,dX$ is the generating function for $\mathcal{L}$.

Examples

To illustrate the variational and Hamiltonian structure of elasticity, some fundamental problems of elasticity, are presented here.

Example 1:Transversal vibration of a beam

Let us investigate the transversal vibration of a beam. The beam is of length *L* and its one end is fixed on a wall, as illustrated in the Figure. Force *f*(*t*) is applied to the other end at time *t*. Let *μ* be the line density of the beam, *E* be its Young's module, and *I* be its geometrical moment of inertia. Let *x* be the distance from the wall and *u*(*x, t*) be the traversal displacement at distance x and time t, as illustrated in the figure. Kinetic energy and bend potential energy of the beam are then described as follows, respectively:



$$T = \int_0^L \frac{1}{2}\mu \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x,$$
$$U = \int_0^L \frac{1}{2}EI \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \mathrm{d}x.$$

Work done by the external force is described as

$$W = f(t).\, u(L, t)$$

We should have

$$\delta\int_{t1}^{t1} L \, dt = 0$$

Which yields the equation of motion of the beam as

$$\mu\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = 0,$$

Subject to the following boundary conditions

$$EI\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0\,,\ u|_{x=0} = 0\,,$$

$$EI\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0, \quad \left.\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)\right|_{x=L} + f = 0.$$

Example 2: small vibrations of a panel flutter (Marsden and Hughes, 1983)

Neglecting nonlinear and two dimensional effects, using the same methos as example 1, for small vibrations of a panel flutter, we will have the following equation

$$\ddot{v} + v'''' - \Gamma v'' + \rho v' = 0 \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = A\begin{pmatrix} v \\ \dot{v} \end{pmatrix} + B\begin{pmatrix} v \\ \dot{v} \end{pmatrix}$$

Where ρ is an aerodynamic pressure, and Γ is an in plane tensile load. If the edges of the plate are simply supported, we impose the boundary conditions $v = 0, v'' = 0$ at x= 0,1. Let

$H_\partial^2 = \left\{ u \in H^2([0,1]) \middle| u = 0 \ at \ x = 0,1 \right\}$ and $X = H_\partial^2 \times L^2$.

Define the operator A on X by

$$A\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{v} \\ -v'''' \end{pmatrix}$$

With $D(A) = \left\{ (v, \dot{v}) \in H_\partial^2 \times L^2 \middle| v \in H^4 = 0, v'' = 0 \ at \ x = 0,1, \dot{v} \in H_\partial^2 \right\}$.

On X define the inner product

$$\langle (v, \dot{v}), (w, \dot{w}) \rangle = \langle v'', w'' \rangle + \langle \dot{v}, \dot{w} \rangle$$

where $\langle \ , \ \rangle$ denotes the L^2 inner product. Let $B : X \to X$ be defined by

$B\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma v'' - \rho v' \end{pmatrix}$, which is a Hamiltonian on X, where the energy is

$H(v, \dot{v}) = \frac{1}{2}\| \dot{v} \|^2 - \frac{\Gamma}{2}\| v' \|^2 + \frac{1}{2}\| v'' \|^2$. B is stable if $0 \le \Gamma \le \pi^2$. (Buckling occurs for $\Gamma > \pi^2$.)

Bifurcation theory and its application to elasticity [Antman, 1995]:

The study of buckling in elasticity is a good example of nonlinear solid mechanics. An important aspect in the study of multiplicity and stability of equilibrium states in bucking is that there are multiple equilibrium states, among which is unbuckled or trivial state.

The simplest model for a planar buckling of a rod is as follows. Suppose that its reference configuration is defined by

$$\mathbf{r}(\mathrm{s}) = \mathrm{s}\,i\,,\ s \in [0,1]$$

We assume that the end s=0 is welded to rigid wall perpendicular to the *i*-axis at the origin 0 and that the end s=1 is free of geometrical restraint and is subject to a compressive force of *P* acting in the *i*-direction. Then the boundary conditions are

$$\mathbf{r}(0) = 0\,,\ \theta(0) = 0\,,\ \mathbf{n}(1) = \text{-}P\,i\,,\ M(1) = 0$$

We assume that no body force or couple is applied to the rod. The integral form of the equilibrium equations is

$$\theta(\mathrm{s}) = \int_0^s \frac{M(\xi)}{EI(\xi)} d\xi\,,\ M(\mathrm{s}) = k.\int_s^1 r_s(\xi)\times n(\xi)\ d\xi = P\int_s^1 \sin\theta(\xi)\ d\xi$$

which can be written as:

$$\frac{d}{ds}\left[\left(EI(S)\right)\theta'(S)\right] + P\sin\theta(s) = 0\,,\ \theta(0) = 0\,,\ \theta'(1) = 0$$

For EI= const., the solution can be found in terms of elliptic functions.

These equations can also be rewritten as:

$$f[P,u] = 0$$

Where u stands for the pair (θ, M). So u is a pair of continuous functions and f is an operator taking such pairs into pairs of continuous functions. The domain of f is a subset of $\chi \times IR^n$ where χ is a real Banach space and that its target is a real Banach space. (

Banach spaces are chosen as they provide a natural way to describe the size and convergence of functions. χ can be the Banach space of continuous functions on a closed and bounded interval.)

We assume that the last equations has enough symmetry that a family of trivial solutions can be readily identified for all values of P and that the variable u is so chosen that all these trivial solutions can be characterized by the equation $u=0$. So we should have $f[P,0]=0,\ \forall P$.

Now the solutions can be represented by bifurcation diagram. Since u is a function, it lies in an infinite-dimensional space, and plotting the solution pairs in (P,u)-space is impossible. Instead, let φ be some convenient real-value function of u, e.g., an amplitude, and plot all points (P,φ) in IR^2 corresponding to solution pairs (P,u). In this example we could take $\varphi(u)$ to be $M(0)$ or $\theta(1)$. A point $(P^0,0)$ on the trivial branch of $f[P,0]=0$ is called a bifurcation point on this branch iff in every neighborhood of this point there is a solution pair (P,u) with $u\neq 0$. Where P^0 is the eigenvalue of the linearization of the integral equation about the trivial branch. If we identify (θ,M) of the integral equation with u in $C^0[0,1]\times C^0[0,1]\equiv\chi$, then its linearization is

$$\theta_1(\mathrm{s})=\int_0^s \frac{M_1(\xi)}{EI(\xi)}d\xi\ ,\ M(\mathrm{s})=P\int_s^1 \theta_1(\xi)\,d\xi$$

Which is equivalent to

$$\frac{d}{ds}\left[\left(EI(S)\right)\theta'_1(S)\right]+P\theta_1(s)=0\,,\ \theta_1(0)=0\,,\ \theta'_1(1)=0$$

It should be noted that, it is NOT true that the behavior of small solutions of $f[P,0]=0$ *is given by its linearization. Bifurcation theory determines what accurate information of* $f[P,0]=0$ *is provided by the linearization.*

Rewrite $f[P,u]$ as

$f[P,u]=u-L(P).u-g[P,u]$

Where $L(P)$ is a linear operator from χ to itself. $L(.)$ is continuous, $g:D\rightarrow\chi$ is continuous, and $g[P,u]=o(\|u\|)$ as $u\rightarrow 0$ uniformly for P in any bounded set. We identify

$$g[p,u](s)=\left(0,P\int_s^1\left[\,\sin\theta(\xi)-\theta(\xi)\right]d\xi\right)$$

With the help of Arzela-Ascoli theorem, one can show that *g, and L* are compact.

<u>Some new works:</u>

Bou-Rabee et al., [2002], numerically examined the stability of a standing cantilever conveying fluid in a multiparameter space. Their numerical bifurcation results obtained from applying the Library of Continuation Algorithms (LOCA) reveal a plethora of one, two, and higher codimension bifurcations.

Champneys, and Fraser (2003) did a comparison between theory and experiment for "Indian wire trick", that is a column longer than its critical length, stabilized by an appropriate vertical vibration at its bottom support, via parametric excitation.

Recently, Argentina and Mahadevan [2005] gave an explanation for the onset of fluid-flow-induced flutter in a flag. Their theory accounts for the various physical mechanisms at work: the finite length and the small but finite bending stiffness of the flag, the unsteadiness of the flow, the added mass effect, and vortex shedding from the trailing edge. They also predicted a critical speed for the onset of flapping as well as the frequency of flapping, and found out that in a particular limit corresponding to a low-density fluid flowing over a soft high-density flag, the flapping instability is akin to a resonance between the mode of oscillation of a rigid pivoted airfoil in a flow and a hinged-free elastic plate vibrating in its lowest mode.

Lew et al. [2004] reviewed and further developed the subject of variational integration algorithms as it applies to mechanical systems of engineering interest. They discussed the conservation properties of both synchronous and asynchronous variational integrators (AVIs). In their work, AVIs are found to result in substantial speed-ups, at equal accuracy, relative to explicit Newmark. In addition, they developed the subject of horizontal variations and configurational forces in discrete dynamics. This theory leads to exact path-independent characterizations of the configurational forces acting on discrete systems. Notable examples are the configurational forces acting on material nodes in a finite element discretisation; and the J-integral at the tip of a crack in a finite element mesh, which are of great importance in solid mechanics.

For a variational and multisymplectic formulation of both continuum mechanics on general Riemannian manifold see Marsden et al. [2001].

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