Geometric Approach to the Motion of a Charged Particle in a Toroidal Magnetic Field

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The motion of a single charged particle in a magnetic field is one of the oldest problems in plasma physics. Such motion typically consists of helical motion around magnetic field lines and a slow drift across field lines. For example, Fig. 1 depicts an electron orbiting a straight current-carrying wire. The wire, not shown, lies along the z-axis, the magnetic field lines wrap around the wire in an angular sense, and the electron drifts both along the field lines in a angular sense and across field lines in the negative z-direction. The main motivation of this write-up is to determine whether this motion can be formulated as a geometric phase. As this is a first attempt, a relatively simple class of magnetic fields with a high degree symmetry will be considered. These magnetic fields support a class of planar particle orbits on which we shall focus; Fig. 2 depicts a planar orbit of an electron orbiting a current-carrying wire. After discussing whether the drift is a geometric phase, we then focus on several other geometric aspects of this motion.

Sec. 1 describes the magnetic fields under consideration. Sec. 2 then attempt to formulate the drift as a geometric phase. The Lagrangian approach does not appear suitable for the motion of paticles in a magnetic field, but we succeed with the phase space approach in the case of planar orbits. In Sec. (3), we introduce the radial action variable and show that various orbit properties can be obtained by differentiating this variable. Sec. (4) applies this principle in regards to the flux enclosed by a trajectory. Finally, in Sec. 5, we apply a technique developed by Montgomery to obtain a relationship between various trajectory properties. In hindsight, this result can be obtained by considering Euler's theorem of homogenous functions.

1 The Magnetic Field and Vector Potential

We consider a restricted class of toroidal magnetic fields which, in cylindrical geometry, have both both axial and azimuthal symmetry:

$$\vec{B} = B(r)\hat{\phi}. \tag{1}$$

Such a magnetic field arises from a current density that is axial and a function of r alone

$$\vec{J} = J(r)\hat{z}. \tag{2}$$

Examples of such situations include straight current-carrying wires and charge-neutralized particle beams.

The vector potential, in magnetostatic situations, is defined by $\nabla \times \vec{A} = \vec{B}$. The ϕ component of this equation is $\partial_z A_r - \partial_r A_z = B_\phi$, from which

$$\vec{A}(r) = A(r)\hat{z} = -\left(\int_{r_0}^{r} B(r')dr'\right)\hat{z}. \tag{3}$$

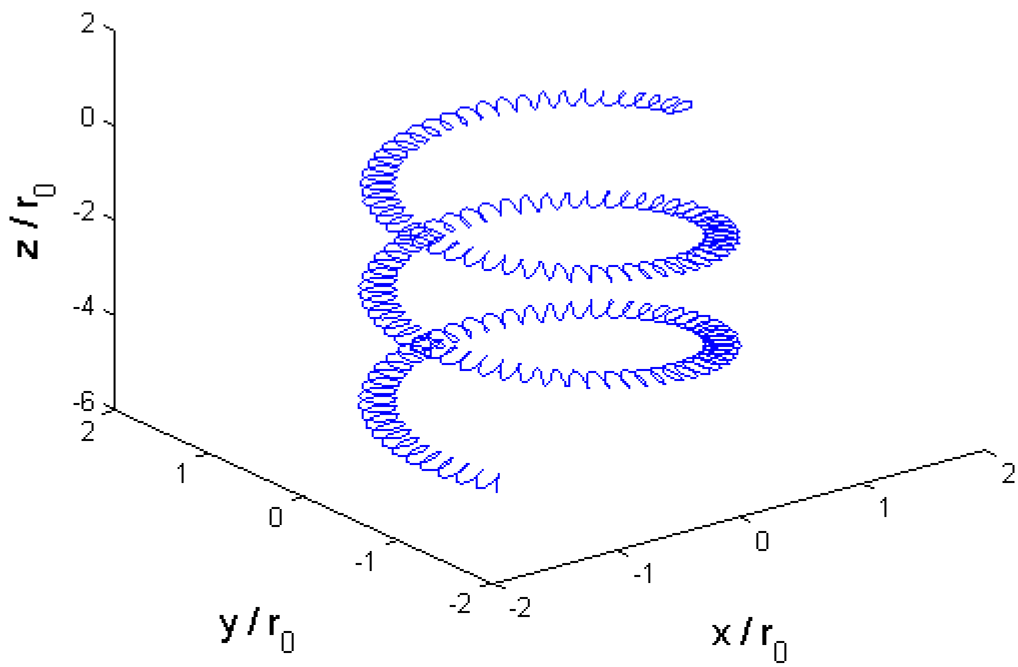

Figure 1: An electron in a toroidal magnetic field executes helical motion around the field lines while slowly drifting in the z-direction.

r_0 is an arbitrary radius which affects $\vec{A}$ only by a constant. Of course, we can add to $\vec{A}$ the gradient of any scalar function, but the gauge of Eq. (3) is actually the most convenient gauge for mechanics since it is independent of z.

2 Reduction Approach to the Drift

The downward drift Δz can be computed using the equation of motion. We seek a geometric approach using the language of symmetry groups and fiber bundles. The Lagrangian approach to reduction introduced difficulties which I could not overcome. The phase space approach, however, admitted a connection in which the entire translation Δz could be considered a geometric phase.

All of the following work is done for planar orbits unless otherwise specified. Our configuration manifold Q will thus be the rz-plane. The symmetry group under consideration is translations in the z-direction. This is a one-dimensional abelian Lie group isomorphic to the real line, and the Lie algebra is likewise one-dimensional. Let ζ be the element of the Lie algebra such that $\zeta_Q = \frac{\partial}{\partial z}$, and let $\mu \in \mathrm{g}^*$ be dual to ζ.

2.1 Difficulties in the Lagrangian Approach

The symmetry group is a symmetry of the configuration space alone, so the problem seems amenable to Lagrangian reduction, which is a reduction of the configuration space. However, I encountered difficulties in this approach.

In Lagrangian reduction, one usually defines a metric on TQ such that the inner product

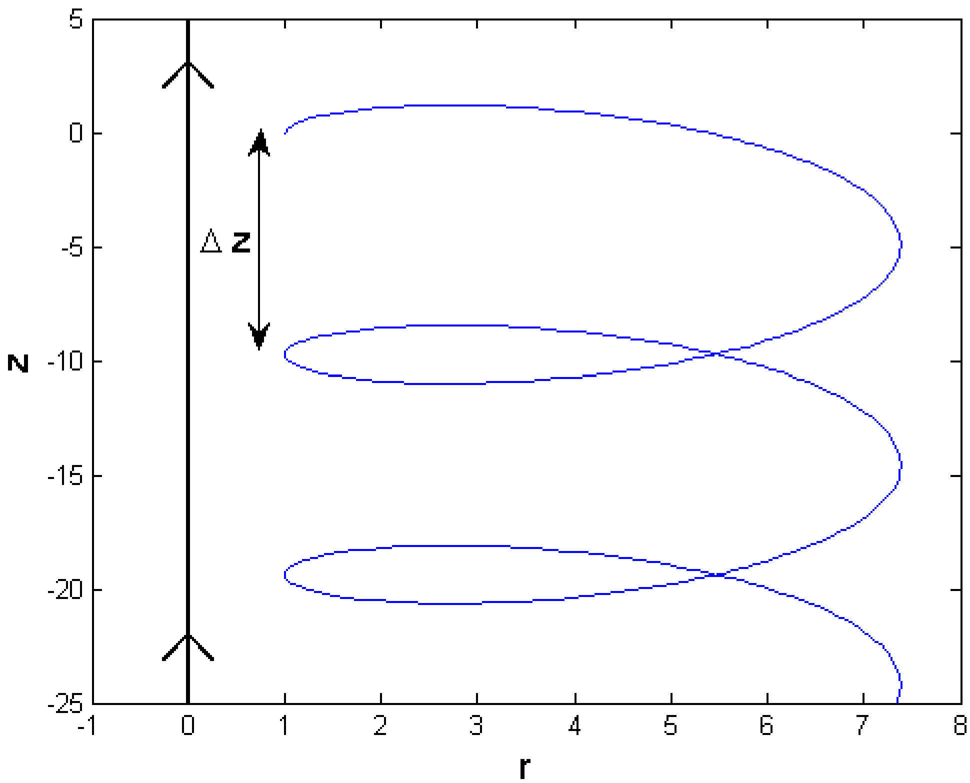

Figure 2: Particle trajectories with zero angular momentum are confined to a plane. The semi-period nature of the orbit is obvious in this diagram, and the axial shift Δz can be formulated as a geometric phase.

of the velocity vector of a trajectory with itself is the kinetic energy. Such a metric is usually constructed via the Legrendre transform:

$$FL \quad : \quad TQ \rightarrow T^*Q \tag{4}$$

$$g(\vec{v},\vec{u}) \quad := \quad \langle FL(\vec{v}),\vec{u}\rangle \tag{5}$$

The metric has to be bilinear and symmetric, meaning the Legendre transform must be self-dual. For the Lagrangian given by Eq. (58), the Legendre transform

$$\vec{P}(\vec{v}) = m\vec{v} + q\vec{A} \tag{6}$$

does not produce a valid metric, as $g(\vec{u},\vec{v}) = \left\langle m\vec{u} + q\vec{A}, \vec{v}\right\rangle$ is neither symmetric nor bilinear.

A similar problem arises when considering the momentum map on Q. The momentum map $\mathbf{J}_L : TQ \rightarrow g^*$ is defined by

$$\langle \mathbf{J}_L(\vec{v}_q), \zeta\rangle = g(\vec{v}_q, \zeta_Q(q)). \tag{7}$$

However, the righthand side is linear in $(\vec{v}_q)_z$ since ζ_Q is in the z-direction. Glancing at the canonical z-momentum defined by Eq. (61), the momentum map of Eq. (7) cannot support the magnetic term, which is not linear in any velocity component.

I cannot successfully incorperate the magnetic field into either the configuration manifold metric nor the configuration manifold momentum map. Accordingly, when constructing the locked inertia tensor and mechanical connection, I can only obtain the trivial versions of these objects.

2.2 Phase-Space Reduction and Magnetic Terms

While the symmetry of the system is indeed a symmetry of the configuration space, the magnetic terms seem to defy the linearity demanded by Lagrangian reduction. We turn to a phase space approach and establish a trivial fiber bundle on a submanifold of constant P_z. By introducing an appropriate connection, we show that the shift Δz can be interpreted as the anholonomy of the bundle.

2.2.1 Phase Space Reduction

Phase space reduction requires a momentum map associated with the symmetry. To reproduce the momentum of Eq. (61), we must treat the magnetic field and vector potential as a two-form and one-form respectively and add them to the canonical two- and one-forms. This is known as a momentum shift [3, p. 176] and, with the Hamiltonian of kinetic energy, reproduces the same equations of motion as Eqs. (63)-(68).

We start by "moving" the vector potential and magnetic field into the canonical one- and two- forms. The two-form version of the magnetic field is defined as $B = i_{\vec{B}}\text{vol}$, where $\text{vol} = rdr \wedge d\phi \wedge dz$. Keeping in mind that $r\hat{\phi}$ is the vector dual to $d\phi$,

$$B = -B(r)dr \wedge dz \tag{8}$$

To compute A, we convert the vector statement $\vec{B} = \nabla \times \vec{A}$ into the equivalent statement for forms: $B = dA$. One solution to this equation is

$$A = A(r)dz. \tag{9}$$

with $A(r)$ defined as in Eq. (3). Of course, we can add the differential of any function we please to A, but we shall stick to this gauge as it is the most convenient for mechanics. We then add these forms to the canonical one- and two-form

$$\begin{aligned}
\Theta &\rightarrow \Theta + A && (10)\\
&= mv_r dr + mv_z dz + qA(r)dz && (11)\\
&= mv_r dr + P_z dz && (12)\\
\Omega &\rightarrow \Omega - B && (13)\\
&= mdr \wedge dv_r + mdz \wedge dv_z + qB(r)dr \wedge dz && (14)
\end{aligned}$$

The magnetic terms in Eqs. (11) and (14) give rise to a non-trivial momentum map. The action of G on phase space is Hamiltonian; that is, we can find a function $J(\zeta)$ whose Hamiltonian vector field is equal to ζ_Q:

$$\begin{aligned}
dJ(\zeta) &= i_{\frac{\partial}{\partial z}}\Omega = mdv_z - qB(r)dr, && (15)\\
J(\zeta) &= mv_z + qA(r). && (16)
\end{aligned}$$

From J, we construct the momentum map $\mathbf{J} : T^*Q \rightarrow g^*$ by stipulating

$$\langle \mathbf{J}(x), \zeta \rangle := J(\zeta)(x). \tag{17}$$

Since μ is dual ζ,

$$\mathbf{J}(x) = (mv_z + qA(r))\,\mu,$$

which agrees with Eq. (61). The phase space momentum map of Eq. (17) contains the magnetic terms that were not supported by the configuration space momentum map given by Eq. (7).

Trajectories lie on submanifolds of constant P_z given by the inverse of the momentum map:

$$\mathbf{J}^{-1}(P_z\mu) = \left\{ \left(r, v_r, z, v_z = \frac{P_z - qA(r)}{m} \right) \in T^*Q \right\}.$$

The submanifold is isomorphic to $\mathbb{R}^3$ and imbeds in T^*Q by chosing v_z such that the momentum is P_z. We then view such submanifolds as trivial fiber bundles with the z-direction as the fiber direction and the rP_r-plane as the base space.

2.3 Reconstruction: Geometric Phases for Planar Orbits

For planar orbits, we can chose a connection form on the total space such that the horizontal lift of the reduced trajectories is the actual unreduced trajectories. Therefore, the anholonomy of the connection is Δz given by Eq. (32), which can thus be considered a geometric phase. Moreover, the connection form depends solely on the magnetic field and not on the energy and z-momentum of the trajectory.

The connection form $\mathbf{A}$ takes values in the Lie algebra and must have the form

$$\mathbf{A} = \alpha \otimes \zeta = [\alpha_r dr + \alpha_{P_r} dP_r + dz] \otimes \zeta \tag{18}$$

where α is a real-valued one-form. The coefficient of unity in front of the dz term is chosen so that $\mathbf{A}(\zeta_Q) = \zeta$. The coefficients α_r and α_{P_r} will be chosen such that the horizontal lift of the reduced trajectory $(r(t), P_r(t))$ is the full trajectory $(r(t), P_r(t), z(t))$. For the curve $(r(t), P_r(t), z(t))$ to be horizontal, its tangent vector must vanish when operated on by the connection. The tangent vector of $(r(t), P_r(t), z(t))$ is the Hamiltonian vector field given by Eqs. (70), (71), and (72):

$$X_H = \frac{P_r}{m}\frac{\partial}{\partial r} - qB(r)\left(P_z - qA(r)\right)\frac{\partial}{\partial P_r} + \frac{P_z - qA(r)}{m}\frac{\partial}{\partial z} \tag{19}$$

Stipulating that $\mathbf{A}[X_H] = 0$ and using Eq. (18),

$$0 \quad = \quad \mathbf{A}\left[X_H\right] = \alpha_r \frac{P_r}{m} - \alpha_{P_r} qB(r)\frac{P_z - qA(r)}{m} + \frac{P_z - qA(r)}{m} \tag{20}$$

This does not completely define $\mathbf{A}$. We further stipulate that the vector $\frac{\partial}{\partial r}$ be a horizontal vector, as the r and z directions are orthogonal in Euclidean space. This imposes $\alpha_r = 0$, and hence

$$\mathbf{A} = \left[\frac{1}{qB(r)} dP_r + dz\right] \otimes \zeta. \tag{21}$$

The curvature two-form is defined by the covariant exterior derivative of the connection form

$$\begin{aligned}
\mathbf{F}\left(v_1, v_2\right) &= d\mathbf{A}\left(\mathrm{hor} v_1, \mathrm{hor} v_2\right) && (22)\\
\mathbf{F} &= -\frac{dB(r)}{qB(r)^2} \wedge dP_r \otimes \zeta && (23)\\
&= -\frac{\partial B}{\partial r}\frac{1}{qB(r)^2} dr \wedge dP_r \otimes \zeta && (24)
\end{aligned}$$

Neither the connection nor the curvature depend on H or P_z, and I believe that this means that the phase of this connection is geometric. It is not difficult to show that this curvature reproduces the well-known "grad-B" drift of the guiding center approximation; see App. (D).

2.4 Attempts to Incorperate Angular Motion

It is tempting to see whether the connection of Eq. (21) can be developed to incorperate non-zero angular momentum. The symmetry group is now the direct product of axial translations and azimuthal rotations. Denote the generator for rotations as η, with $\eta_Q = \frac{\partial}{\partial\phi}$. In this case, the general form of the connection is

$$\begin{aligned}
\mathbf{A} &= \alpha \otimes \zeta + \beta \otimes \eta && (25)\\
&= [\alpha_r dr + \alpha_{P_r} dP_r + dz] \otimes \zeta + [\beta_r dr + \beta_{P_r} dP_r + d\phi] \otimes \eta && (26)
\end{aligned}$$

so that $\mathbf{A}[\zeta_Q] = \zeta$ and $\mathbf{A}[\eta_Q] = \eta$.

If we demand once again that the full trajectory be horizonal and $\alpha_r = \beta_r = 0$, we would have

$$\begin{aligned}
\alpha_{P_r} &= -\frac{P_z - qA}{P_\phi^2/r^3 - qB(P_z - qA)} && (27)\\
\beta_{P_r} &= --\frac{P_\phi/r^3}{P_\phi^2/r^3 - qB(P_z - qA)} && (28)
\end{aligned}$$

These connection coefficients are not as pretty as Eq. (21), although they reduce to Eq. (21) in the case $P_\phi = 0$. They also contain the conserved momenta P_z and P_ϕ. I do not know if this implies that the phase of such a connection is not geometric, as the connection depends on the trajectory parameters.

3 The Radial Action Variable

The motion of a charged particle in the magnetic fields given by Eq. (1) is semi-periodic. The particle returns to its initial starting radius with its initial velocity but at a different axial and angular position. Put another way, the projection of the phase space trajectory onto the rP_r-plane is closed, even though the full phase space trajectory and the trajectory in physical space are not. This claim has the implicit assumption that such orbits have two radial turning points; this is not generically true and must be checked by attempting to solve the equation

$$P_r = 0 = \pm\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2}. \tag{29}$$

Since the trajectory is closed in the rP_r-plane, we can define the radial action variable J as the area enclosed by the trajectory in the rP_r-plane. This is typically expressed as an integral over r:

$$J(H, P_z, P_\phi) = \int\int dr dP_r = \oint P_r dr = \oint \pm\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2} dr \tag{30}$$

For planar orbits, we set $P_\phi = 0$

$$J(H, P_z) = J(H, P_z, P_\phi = 0) = \oint \pm\sqrt{2mH - (P_z - qA(r))^2} dr \tag{31}$$

We denote the period of motion as Δt, the distance traveled in the z-direction as Δz, and the change in azimuthal angle by $\Delta\phi$. Such quantities characterize the average motion of the particle; for instance, the drift velocity in the z direction is $\Delta z/\Delta t = \lim_{t\to\infty} z(t)/t$. These quantities can be computed from Eqs. (63)-(66):

$$\begin{aligned}
\Delta z &= \oint dz = \oint \frac{\dot{z}}{\dot{r}} dr = \oint \frac{P_z - qA(r)}{\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2}} dr = -\frac{\partial J}{\partial P_z}, && (32)\\
\Delta t &= \oint dt = \oint \frac{1}{\dot{r}} dr = \oint \frac{m}{\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2}} dr = \frac{\partial J}{\partial H}, && (33)\\
\Delta \phi &= \oint d\phi = \oint \frac{\dot{\phi}}{\dot{r}} dr = \oint \frac{P_\phi/mr^2}{\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2}} dr = -\frac{\partial J}{\partial P_\phi}. && (34)
\end{aligned}$$

The final equalities of each line can be verified by direct differentation of Eq. (30) or more generally (using P_z as an example),

$$\frac{\partial J}{\partial P_z} = \oint \frac{\partial P_r}{\partial P_z} dr = \oint -\left(\frac{\partial P_r}{\partial H}\right)_{P_z} \left(\frac{\partial H}{\partial P_z}\right)_{P_r} dr = \oint -\frac{\dot{z}}{\dot{r}} dr = -\Delta z. \tag{35}$$

This second proof is independent of the particular Hamiltonian and shows that the shift in a cyclic variable over one period is given by differentating the action variable with respect to the associated conserved momentum. If the Hamiltonian included additional cyclic coordinates, their shifts over a radial excursion would be given by analogous derivatives. It is a well-known fact that differentiating J with respect to H yields the period of motion [1, p.461], but I cannot find any mention of the above generalization in any classic text. The only example I have found is for the toroidal drift in a axisymmetric magnetic field [2], and the principal is not proved in general there.

4 Flux Enclosed by an Orbit

The magnetic flux enclosed by a trajectory is of interest because, when the magnetic field strength changes in time, the particle's energy increases in proportion to the change in flux. The calculation is surprising delicate and can be found in App. (C); the result is:

$$\Phi = \frac{\partial J}{\partial q} - \frac{1}{q}\left(P_z - m\frac{\Delta z}{\Delta t}\right)\Delta z \tag{36}$$

The second term tends to vanish for Larmor-like (e.g. nearly circular) orbits, in which case the flux is given by the partial derivative of J with respect to charge q.

Kaluza-Klein theory postulates that q is a conserved canonical momentum conjugate to a cyclic variable. The presence of the term $\partial J/\partial q$ in Eq. (36) is curious in light of Eq. (35). Could magnetic flux be the cyclic coordinate associated with q? The possibility is spoiled by the presence of the additional terms in Eq. (36). I have attempted to formulate Eq. (36) as a covariant derivative of J,

$$\Phi = \nabla J = \frac{\partial J}{\partial q} + \Gamma J \tag{37}$$

This would require the connection coefficient to

$$\Gamma = -\frac{1}{q}\left(P_z - m\frac{\Delta z}{\Delta t}\right)\frac{\Delta z}{J}, \tag{38}$$

which looks rather strange and has no natural explanation or motivation.

We could define a cyclic coordinate χ conjugate to q such that the change in χ over one gyration is $\partial J/\partial q$.

$$\Delta\chi = \frac{\partial J}{\partial q} = \oint \vec{A}\cdot d\vec{l} \tag{39}$$

$$\dot{\chi} = -\frac{\partial H}{\partial q} = A\frac{P_z - qA}{m} \tag{40}$$

χ is the "magnetic action" (the contribution to the action from the magnetic term $q\vec{v}\cdot\vec{A}$) divided by q; we can see this by integrating the Lagrangian over one period of motion:

$$\Delta S = \oint L dt \tag{41}$$

$$= \frac{1}{2}mv^2\Delta t + q\Delta\chi \tag{42}$$

I am still exploring the relationship bewteen χ and Φ; I suspect the two might be related by changing from the lab frame to the drift frame.

5 Montgomery's Circuit and Euler's Theorem

Montgomery derived a formula for the phase acquired by a rigid body by considering the integral of the canonical one-form over a special contour in phase space [4]. By considering an analogous contour, we arrive at Euler's theorem of homogenous functions for the action variable and a formula that relates the action variable to the flux, energy, and period of an orbit.

Montgomery considered the following contour. The contour first follows the trajectory over one 'period' of motion: the body angular momentum has returned to its original value but the whole body has undergone a net rotation. The second half of the contour travels back to the starting point by 'unrotating' the body. This second step does not follow the equations of motion but instead travels fiberwise. Montgomery then integrates the canonical one-form along this path and invokes Stoke's theorem to obtains the formula for the geometric phase.

By analogy, we consider a contour that follows the trajectory over one gyration and then travels in the z-direction to rejoin the starting point. We will integrate the one-form θ over this contour. For the first part, kinetic energy is constant, so

$$\int_I \theta = \int_I [mv_r dr + mv_z dz + +qA(r)dz] \tag{43}$$

$$= \int_I m\left[v_r^2 + v_z^2\right] dt + \int_I qA(r)dz \tag{44}$$

$$= 2H\Delta t + q\frac{\partial J}{\partial q} \tag{45}$$

We now integrate over the second branch, for which r is constant.

$$\int_{II} \theta = \int_{II} [mv_r dr + mv_z dz + qA(r)dz] \tag{46}$$

$$= \int_{II} [mv_z dz + qA(r)dz] \tag{47}$$

$$= \int_{II} P_z dz = -P_z\Delta z \tag{48}$$

The minus sign occurs because we are travelling backwards (in the sense opposite to Δz). Since the total contour is closed, we apply Stoke's theorem. The contour lies entirely on a submanifold of constant P_z, so $d\left(mv_z dz + qA(r)dz\right) = d\left(P_z dz\right) = 0$. We are left with

$$\int d\theta = \int mdr \wedge dv_r \quad = \quad J(H, P_z) \tag{49}$$

Equating the line integral, Eqs. (45) and (48) and the surface integral given by Eq. (49) gives,

$$J(H, P_z, q) \quad = \quad 2H\Delta t + q\frac{\partial J}{\partial q} - P_z\Delta z \tag{50}$$

$$\quad = \quad mv^2\frac{\partial J}{\partial H} + \beta\frac{\partial J}{\partial \beta} + P_z\frac{\partial J}{\partial P_z} \tag{51}$$

$$J(v, P_z, q) \quad = \quad v\frac{\partial J}{\partial v} + q\frac{\partial J}{\partial q} + P_z\frac{\partial J}{\partial P_z} \tag{52}$$

We could have arrived at this formula by Euler's theorem of homogenous functions, for J is indeed a homogenous function of degree one when we take its variables to be v, β, and P_z.

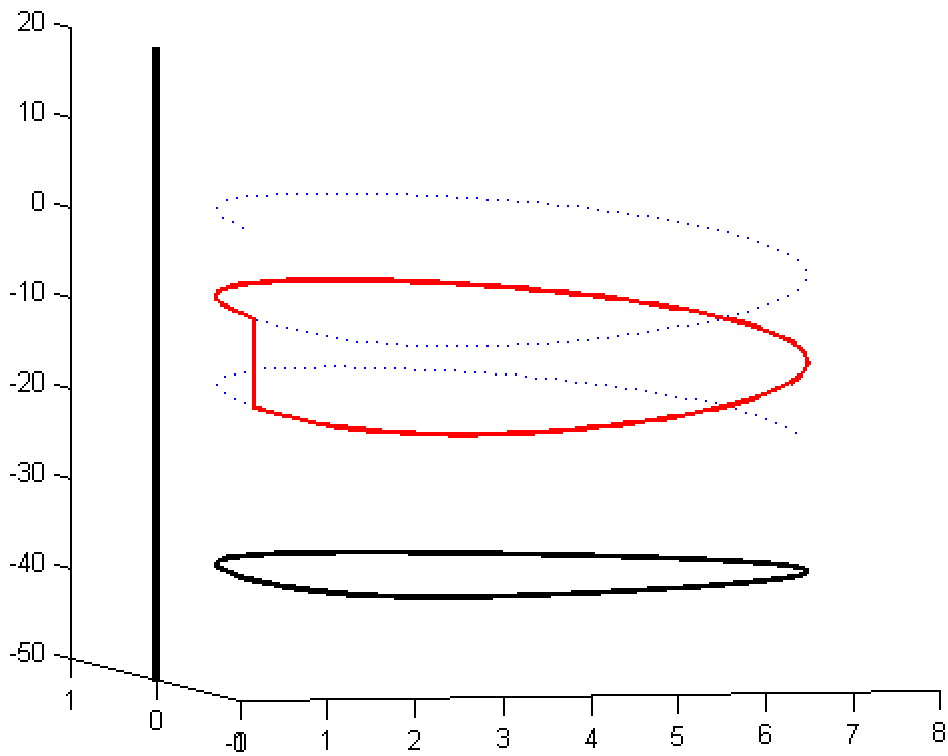

Figure 3: The phase space contour under consideration.

The flux enclosed by a gyro-orbit is given by Eq. (36), rewritten here:

$$q\Phi = q\frac{\partial J}{\partial q} + P_z\frac{\partial J}{\partial P_z} + mv_d^2\frac{\partial J}{\partial H}. \tag{53}$$

Using Eq. (53) in Eq. (52),

$$J \quad = \quad \Phi + 2\left(H - \frac{1}{2}mv_d^2\right)\Delta t \tag{54}$$

In the guiding center limit, J becomes proportional to the first adiabatic invariant μ, and the drift velocity energy is negligible to the total energy H, so we get

$$-2\pi\frac{m}{q}\mu = \Phi + 2H\Delta t \tag{55}$$

6 Conclusions and Directions for Future Work

The Lagrangian approach to reduction could not be completed, but the phase space approach successfully produced a connection for planar orbits such that Δz is a geometric phase. This procedure can be extended to include angular motion as well, but I am not sure whether the anholonomies are purely geometric. All of the averaged orbit properties such as Δt, Δz and Φ can be derived from partial derivatives of J. The expression for Φ contains a partial derivative of J with respect to q, which is reminiscent of Kaluza-Klein theory. Finally, by considering a phase space contour akin to that considered by Montgomery, a new relationship can be derived for orbit properties.

The following are several ideas on how to pursue the subject further:

- Is the connection derived for angular motion geometric? Can we just keep defining whatever connection is needed to make the phase geometric?

- The Lagrangian approach failed because the magnetic terms could not be incorperated into a configuration manifold metric. If, however, we follow Kaluza-Klein theory and allow an extra dimension, then the Hamiltonian in Eq. (62) can be written in metric form:

$$H = \frac{1}{2m}\begin{pmatrix} P_r & P_z & q \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 0 \end{pmatrix}\begin{pmatrix} P_r \\ P_z \\ q \end{pmatrix}, \tag{56}$$

  and the same can be done with the Lagrangian. Perhaps, then, one could formulate a Lagrangian approach starting from the higher dimensional space. Maybe the order of reduction is significant?

- Why is Φ is so much more difficult to calculate than other orbit properties such as Δz or Δt?

- What is the connection between the flux Φ and $\partial J/\partial q$? I suspect that they might be related by a canonical transform from the lab frame to the drift frame.

- We were able to obtain the flux Φ in two different ways: one by averaging, and again by changing to the drift frame. Is there a connection between the two?

- *Eq.* (53) seems to me to be a reshuffling of terms that go like canonical momentum times shift is conjugate coordinate. Am I really just unknowingly re-expressing the canonical one-form in a new set of variables?

A Standard Analysis of Motion

In this section, we employ the usual approach to analyzing particle motion in a magnetic field. We write down the Lagrangian, canonical momenta, Hamiltonian, and equations of motion and show the existence of a planar orbits.

A.1 Lagrangian and Hamiltonian

The Lagrangian for charged particles in a magnetic field described by Eq. (1) is

$$
\begin{aligned}
L &= \frac{1}{2}mv^2 + q\vec{v}\cdot\vec{A} && (57)\\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + qA(r)\dot{z}, && (58)
\end{aligned}
$$

and the canonical momenta are

$$
\begin{aligned}
P_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} && (59)\\
P_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} && (60)\\
P_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA(r). && (61)
\end{aligned}
$$

Since the Lagrangian is independent of z and ϕ, both P_z and P_ϕ are conserved along trajectories.

The Hamiltonian is obtained by the usual Legendre transform and is just the kinetic energy expressed in terms of the canonical momenta:

$$
H = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} + \frac{(P_z - qA(r))^2}{2m}. \qquad (62)
$$

This Hamiltonian is independent of time and thus is conserved along trajectories.

A.2 Phase Space Equations of Motion

The phase space equations of motion follow from Hamilton's equations. Since H, P_ϕ, and P_z are all conserved along a trajectory, we can solve Eq. (62) for P_r in terms of r and these conserved quantities.

$$
\begin{aligned}
\dot{\phi} &= \frac{P_\phi}{mr^2} && (63)\\
\dot{z} &= \frac{P_z - qA(r)}{m} && (64)\\
\dot{r} &= \frac{P_r}{m} && (65)\\
&= \pm\frac{\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2}}{m} && (66)\\
\dot{P}_r &= -\frac{\partial H}{\partial r} = \frac{P_\phi^2}{mr^3} + \frac{P_z - qA(r)}{m}q\frac{dA}{dr} && (67)\\
&= \frac{P_\phi^2}{mr^3} - qB(r)\frac{P_z - qA(r)}{m} && (68)
\end{aligned}
$$

Most notable is that fact that all these velocities depend only on r and conserved quantities. Therefore, should the particle return to a particular radius, it will have the same velocity it originally had at that radius. This gives rise to the semi-periodic nature of the orit.

A.3 Planar Orbits

The magnetic fields described by Eq. (1) support a class of planar orbits. These orbits have $P_\phi = 0$, so that, by Eq. (63), $\dot{\phi} = 0$. There is no angular motion, and the trajectory is confined to a plane containing the z-axis. The Hamiltonian and equations of motion for such orbits are obtained by setting $P_\phi = 0$ above:

$$\begin{aligned} H &= \frac{P_r^2}{2m} + \frac{(P_z - qA(r))^2}{2m} && (69) \\ \dot{z} &= \frac{P_z - qA(r)}{m} && (70) \\ \dot{r} &= \frac{P_r}{m} = \pm\frac{\sqrt{2mH - P_\phi^2/r^2 - (P_z - qA(r))^2}}{m} && (71) \\ \dot{P}_r &= qB(r)\frac{P_z - qA(r)}{m} && (72) \end{aligned}$$

B Relationship between J and μ

Our goal is to show that the radial action variable J reduces to the first adiabatic invariant μ [5, p. 16] in the guiding center approximation. J is the signed area of projection of the trajectory into the rP_r-plane and is usually written as

$$J = \oint P_r dr = \oint \pm\sqrt{2mH - (P_z - qA(r))^2} dr. \qquad (73)$$

This form does not admit a useful expansion of the integrand, as $(P_z - qA(r))^2$ varies from 0 to $2mH$ over the range of integration. We shall take a different tack and integrate over P_r instead.

$$J = \oint P_r dr = -\oint r dP_r = -\oint A^{-1}\left(\frac{1}{q}P_z - \frac{1}{q}\sqrt{2mH - P_r^2}\right) dP_r \qquad (74)$$

where we have solved Eq. (69) for r as a function of P_r. A^{-1} represents the inverse function of the vector potential. We can perform a Taylor expansion of the integrand. Let $r_{\text{gc}} = A^{-1}(P_z/q)$; this is the guiding center radius (center of the gyro-orbit) and is, for these magnetic fields, a constant of motion. Then

$$\begin{aligned} J &\approx -\oint \left[A^{-1}\left(\frac{1}{q}P_z\right) - \left(\frac{1}{dA/dr}\right)\frac{1}{q}\sqrt{2mH - P_r^2}\right] dP_r && (75) \\ &= -\oint \left[r_{\text{gc}} - \left(-\frac{1}{B(r_{\text{gc}})}\right)\frac{1}{q}\sqrt{2mH - P_r^2}\right] dP_r && (76) \\ &= 0 - \frac{\pi m^2 v^2}{qB(r_{\text{gc}})} = -2\pi\frac{m}{q}\left(\frac{mv^2}{2B(r_{\text{gc}})}\right) = -2\pi\frac{m}{q}\mu. && (77) \end{aligned}$$

The r_{gc} term vanishes as the integral of a constant over a closed contour. The second integral is evaluated by identifying it as the area enclosed by a circle of area $\pi(mv)^2$.

C Magnetic Flux

In this section, we derive Eq. (36) for the flux enclosed by one gyration of motion. A priori, the flux appears to be ill-defined: the trajectory does not fully close on itself, so there is no definite surface through which to measure flux. There is, however, a chosen 'correct' value of flux, for if one changes the magnetic field strength in time, the particle's kinetic energy changes in a way that depends on the flux enclosed by a gyration. This value of flux, derived here, yields theoretical predictions that agree with numerical simulations of a particle in a time-dependent magnetic field.

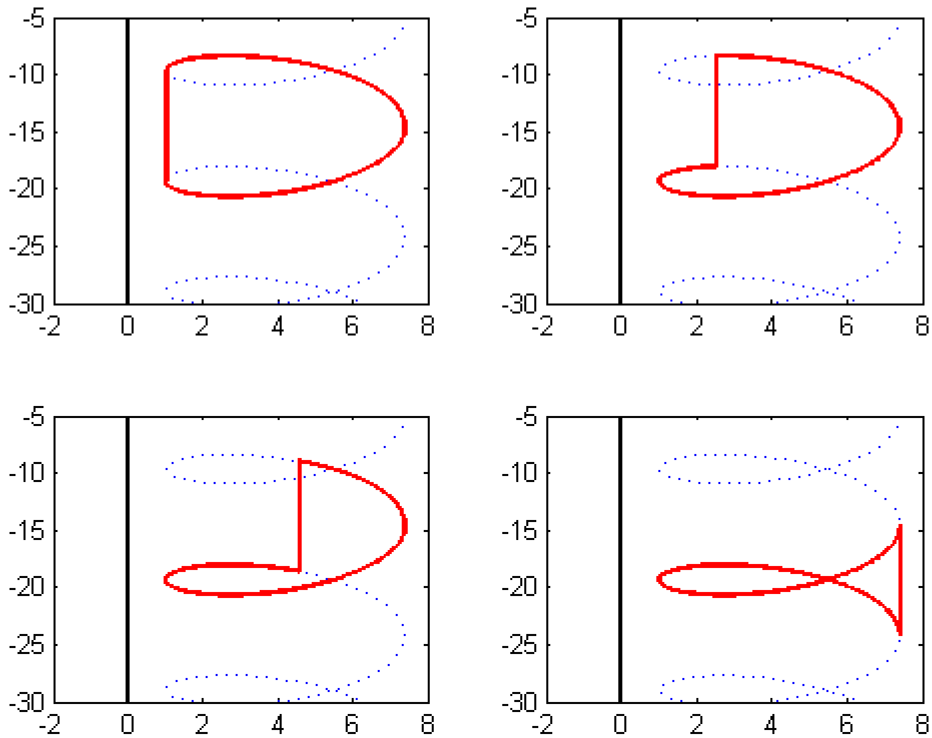

Figure 4: The flux through a gyro-orbit depends on where one starts the gyration.

To start, one could compute the flux by artificially close the trajectory by connecting the end points with a straight line in the z-direction with length Δz. If we begin the gyration at radius r_0, then the connecting path goes from $r = r_0$, $z = z_0$ to $r = r_0$, $z = z_0 + \Delta z$, and the flux is

$$\begin{aligned}
\Phi &= \oint \vec{A} \cdot dl = \oint_{\text{trajectory}} \vec{A} \cdot dl + \oint_{\text{connecting}} \vec{A} \cdot dl & (78)\\
&= \oint_{\text{trajectory}} A(r)dz - A(r_0)\Delta z & (79)\\
&= \oint A(r) \frac{P_z - qA(r)}{\sqrt{2mH - (P_z - qA(r))^2}} dr - A(r_0)\Delta z & (80)\\
&= \frac{\partial J}{\partial q} - A(r_0)\Delta z & (81)
\end{aligned}$$

This flux clearly depends on r_0, and r_0 is arbitrary. Any of my attempts to chose a 'special' r_0 does not give agreement with numerical simulations.

To derive Eq. (flux) from Eq. (81), one has to average Eq. (81). Noting that $qA(r) = P_z - m\dot{z}$ and that the average of $\dot{z}$ is $\Delta z/\Delta t$, we have

$$\Phi = \frac{\partial J}{\partial q} - \frac{1}{q}\left(P_z - m\frac{\Delta z}{\Delta t}\right)\Delta z \tag{82}$$

Alternatively, one could move to the drift frame in which the orbit is closed. If the transformation is done properly, the flux integral evaluates to Eq. (36). It is this expression that agrees with numerical simulations of a particle in a changing magnetic field.

D Relating Curvature to the Grad-B Drift

In the standard guiding center approximation, the grad-B drift estimates the drift velocity of a particle in a magnetic field with a gradient in the field strength. The expression for the grad-B drift is

$$\vec{v}_{\nabla B} = -\frac{mv_L^2}{2qB^3}\nabla B \times \vec{B}. \tag{83}$$

For the magnetic fields we are considering, ∇B is in the r-direction while $\vec{B}$ is in the ϕ-direction. Also, the Larmor speed v_L is the full trajectory speed v. This gives

$$v_{\nabla B} \quad = \quad -\frac{mv^2}{2qB^2}\frac{\partial B}{\partial r}\hat{z}. \tag{84}$$

This formula only applies to trajectories that resemble Larmor (i.e. nearly circular) orbits. For such trajectories, we estimate the quantity Δz by integrating the curvature form. We assume that the change in the field is small over such an orbit and treat the magnetic field and its gradient as constants over the integration:

$$\begin{aligned}
\Delta z &= \int -\frac{\partial B}{\partial r}\frac{1}{qB^2}dr \wedge dP_r && (85)\\
&\approx -\frac{\partial B}{\partial r}\frac{1}{qB(r_{\text{gc}})^2}\int dr \wedge dP_r && (86)\\
&= -\frac{\partial B}{\partial r}\frac{1}{qB(r_{\text{gc}})^2}J && (87)
\end{aligned}$$

We now use Eq. (77) to estimate J. We also assume that the angular frequency is given by the cyclotron frequency $\omega_c = qB(r_{\text{gc}})/m$, so the period is $\Delta t = 2\pi m/(qB(r_{\text{gc}}))$. When we put it all together,

$$\begin{aligned}
\frac{\Delta z}{\Delta t} &\approx \left(-\frac{\partial B}{\partial r}\frac{1}{qB(r_{gc})^2}J\right)\cdot\left(\frac{2\pi m}{qB(r_{gc}))}\right)^{-1} && (88)\\
&= \left(-\frac{\partial B}{\partial r}\frac{1}{qB(r_{gc})^2}\frac{\pi m^2 v^2}{qB(r_{gc})}\right)\cdot\left(\frac{qB(r_{gc})}{2\pi m}\right) && (89)\\
&= -\frac{\partial B}{\partial r}\frac{1}{qB(r_{\text{gc}})^2}\frac{1}{2}mv^2 && (90)
\end{aligned}$$

which agrees with Eq. (84).

Interestingly, for a vacuum field $B(r) = \mu_0 I/(2\pi r)$, the curvature form is constant, and the approximation of pulling the magnetic field terms out of the integral is actually exact. Δz is then proportional to J for all orbits, not just Larmor-like ones. Since $\Delta z = -\partial J/\partial P_z$, we could guess that J is an exponential function of P_z, which is indeed that case if one evaluates the integral for J.

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