Controlled Lagrangians and the Inverted Pendulum on a Cart

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1 Introduction

Examples of systems in unstable equilibrium, also known as balance systems, are ubiquitous – an upright person is a balance system, as well as bicycles or even something as silly as balancing a broom on one's palm – and the need to control balance systems to prevent them from leaving equilibrium due to small changes in the environment, or perturbations, is an example where closed-loop feedback is used. Closed-loop feedback is where a system is driven, the result of the driving is monitored and then changes of the input are made accordingly. (This is exactly what is done when driving a car – the steering wheel is controlled, the driver sees the results of the turning and makes adjustments as needed in order to get the desired result.)

In this paper I will describe the method of Bloch, Leonard and Marsden [1][2] of *controlled Lagrangians* where a mechanical system with symmetry is stabilized by modifying the uncontrolled (free) Lagrangian by applying control forces to certain variables of the Lagrangian which then stabilize other variables. This is done in such a way that the closed-loop dynamics of the controlled system is still in Lagrangian form. This is desirable because a Liapunov function can then be used to determine the necessary control gains to achieve stability. This method is particularly nice because the Euler-Lagrange equations from the controlled Lagrangian will include new terms that can be associated with the control forces.

After describing the general method I will then apply it to the inverted pendulum on a cart, where by controlling the cart, the pendulum can be stabilized in the upright position.

2 Background

The configuration space Q of a mechanical system is a space where every state of the system, i.e. every allowed combination of the independent generalized coordinates, is represented by a point. For example, the generalized coordinates of the inverted pendulum on a cart are θ, the angle the pendulum makes with the vertical, and s, the horizontal displacement of the cart. (Note l, the length of the pendulum arm is constant.) Thus Q for the inverted pendulum on a cart is $Q = S^1 \times \mathbb{R}$. In this paper I will assume that $Q = S \times G$ where G is a Lie group, so that by controlling variables in G we wish to control variables in S.

First, note that kinetic energy can be written using a metric tensor, $T = \frac{1}{2}g(\dot{q}, \dot{q})$. This metric tensor can also be more generally written as $g(v, v) = g(\text{Hor}v, \text{Hor}v) + g(\text{Ver}v, \text{Ver}v)$, where Ver v is tangent to the orbits of G acting on Q and Hor v is chosen to make the new expression for the metric consistent (i.e. $g(\text{Hor}v, \text{Ver}v) = 0$). Ver v can be thought of as the piece of v in the direction of the action of G and Hor v as the piece "metric orthogonal" to the direction of the action, so the horizontal and vertical spaces are g-orthogonal.

The Lagrangian of the system is then modified by defining a new choice of horizontal space denoted Hor_τ and changing metric on horizontal vectors ($g \to g_\sigma$) and vertical vectors ($g \to g_\rho$). This is accomplished by letting τ be a one form on Q that annihilates vertical vectors and defining $\text{Hor}_\tau v_q = \text{Hor}v_q - [\tau(v)]_Q(q)$ and $\text{Ver}_\tau v_q = \text{Ver}v_q + [\tau(v)]_Q(q)$.

Given τ, σ and ρ we then define the controlled Lagrangian to be

$$L_{\tau,\sigma,\rho}(v) = \frac{1}{2}[g_\sigma(Hor_\tau v_q, Hor_\tau v_q) + g_\rho(Ver_\tau v_q, Ver_\tau v_q] - V(q) \qquad (1)$$

where V is the potential energy of the system.

This controlled Lagrangian is then compared to the original Lagrangian of the system with the control law added and the two Lagrangians are matched. The feedback law for u is then solved for and finally the stability of an equilibrium is determined.

3 The Inverted Pendulum on a Cart

We will now apply the method of controlled Lagrangians to the inverted pendulum on a cart. The position vector for the pendulum is $\vec{x}_{\text{pend}} = (s + l\sin\theta, l\cos\theta)$ giving a velocity vector of $\vec{v}_{\text{pend}} = (\dot{s} + l\cos\theta\dot{\theta}, -l\sin\theta\dot{\theta})$. The velocity of the cart is simply $\vec{v}_{\text{cart}} = (\dot{s}, 0)$ so the kinetic energy of the pendulum cart system is just $\frac{1}{2}(mv_{\text{pend}}^2 + Mv_{\text{cart}}^2)$ where m is the mass of the pendulum and M is the mass of the cart.

The Lagrangian for the system is $L = T - V$, where V is the potential energy, i.e. $V = mgl\cos\theta$ yielding a Lagrangian of

$$L(\theta, s, \dot{\theta}, \dot{s}) = \frac{1}{2}(m + M)\dot{s}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + 2\dot{s}l\cos\theta\dot{\theta}) - mgl\cos\theta \qquad (2)$$

Let $\alpha = ml^2, \beta = ml, \gamma = M + m$ and $D = -mgl$ so we have

$$L(\theta, s, \dot{\theta}, \dot{s}) = \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta\dot{s}\cos\theta\dot{\theta}) + D\cos\theta \qquad (3)$$

The equations of motion for the system with a control force u acting on the cart are given by the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} &= u. \end{aligned} \tag{4}$$

Note that s is a cyclic variable so $\frac{\partial L}{\partial s} = 0$ and we have

$$\alpha\ddot{\theta} - \beta \sin\theta \dot{\theta}\dot{s} + \beta\cos\theta\ddot{s} + \beta\sin\theta\dot{s}\dot{\theta} + D\sin\theta = 0 \tag{5}$$

and

$$\gamma\ddot{s} - \beta\sin\theta\dot{\theta}^2 + \beta\cos\theta\ddot{\theta} = u \tag{6}$$

Now we need to form the controlled Lagrangian. For our example we will take $g_\rho = g$, $g_\sigma = \sigma g$ when acting on horizontal vectors. Using these assumptions, let's prove the following theorem:

Theorem 1 If $g_\rho = g$, $g_\sigma = \sigma g$ when acting on horizontal vectors then the control Lagrangian reduces to

$$L_{\tau,\sigma,\rho}(v) = L(v + \tau(v)_Q) + \frac{\sigma}{2} g(\tau(v)_Q, \tau(v)_Q) \tag{7}$$

Proof. Let's look at the first term of the control Lagrangian

$$\begin{aligned} &\frac{1}{2} g_\sigma(Hor_\tau v_q, Hor_\tau v_q) \\ &= \frac{1}{2} g_\sigma(Hor(v) - \tau(v)_Q, Hor(v) - \tau(v)_Q) \\ &= \frac{1}{2}[g(Hor(v), Hor(v)) + \sigma g(\tau(v)_Q, \tau(v)_Q)] \end{aligned}$$

For the second term we have

$$\begin{aligned} &\frac{1}{2} g(Ver(v) + \tau(v)_Q, Ver(v) + \tau(v)_Q) \\ &= \frac{1}{2} g(Ver(v), Ver(v)) + g(Ver(v), \tau(v)_Q) + \frac{1}{2} g(\tau(v)_Q, \tau(v)_Q) \\ &= \frac{1}{2} g(Ver(v), Ver(v)) + g(v, \tau(v)_Q) + \frac{1}{2} g(\tau(v)_Q, \tau(v)_Q) \end{aligned}$$

Note that the last line follows from $v =$Hor$(v)+$Ver(v) and $g(\text{Hor}(v), \text{Ver}(v)) = 0$ so $g(v, \tau(v)_Q) = g(\text{Hor}(v), \tau(v)_Q)) + g(\text{Ver}(v), \tau(v)_Q) = g(\text{Ver}(v), \tau(v)_Q$, because $g(\text{Hor}(v), \tau(v)_Q) = 0$.

Finally adding the expressions for the two terms and subtracting the potential gives

$$\frac{1}{2}[g(Hor(v)+Ver(v),Hor(v)+Ver(v))+g(v,\tau(v)_Q)+\frac{1}{2}g(\tau(v)_Q,\tau(v)_Q)+\sigma g(\tau(v)_Q,\tau(v)_Q)]-V(q)$$
$$=\frac{1}{2}g(v,v)+g(v,\tau(v)_Q)+\frac{1}{2}g(\tau(v)_Q,\tau(v)_Q)+\sigma g(\tau(v)_Q,\tau(v)_Q)]-V(q)$$
$$=\frac{1}{2}g(v+\tau(v)_Q,v+\tau(v)_Q)+\sigma g(\tau(v)_Q,\tau(v)_Q)]-V(q)$$

the desired expression.

We want τ to be a one-form on G such that $\tau[\text{Ver}(v)]=0$. In the case of the inverted pendulum on a cart the most general form τ can take is $\tau=k(\theta)d\theta$. Note that $\tau(v)_Q=k\dot{\theta}$. Plugging this into our simplified expression for the control Lagrangian yields

$$L_{\tau,\sigma}=\frac{1}{2}(\alpha\dot{\theta}^2+2\beta\cos\theta(\dot{s}+k\dot{\theta})\dot{\theta}+\gamma(\dot{s}+k\dot{\theta})^2+\frac{\sigma}{2}\gamma k^2\dot{\theta}^2+D\cos\theta \tag{8}$$

The Euler-Lagrange equations are then

$$\begin{aligned}&\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial\dot{s}}-\frac{\partial L}{\partial s}\\&=\frac{d}{dt}(\beta\cos\theta\dot{\theta}+\gamma(\dot{s}+k\dot{\theta})\\&=-\beta\sin\theta\dot{\theta}^2+\beta\cos\theta\ddot{\theta}+\gamma(\ddot{s}+k'\dot{\theta}^2+k\ddot{\theta})\end{aligned} \tag{9}$$

$$\begin{aligned}&\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}\\&\frac{d}{dt}(\alpha\dot{\theta}+\beta\cos\theta\dot{s}+2k\beta\cos\theta\dot{\theta}+k\gamma(\dot{s}+k\dot{\theta})+\sigma\gamma k^2\dot{\theta})-\end{aligned}$$

Comparing the two s equations gives $u=-\gamma k'\dot{\theta}^2-\gamma k\ddot{\theta}$. Using this expression for u and solving the s equation for $\ddot{s}$ gives $\ddot{s}=\frac{\beta}{\gamma}(\sin\theta\dot{\theta}^2-\cos\theta\ddot{\theta})-k'\dot{\theta}^2-k\ddot{\theta}$. Substituting this into the θ equation gives at last the controlled Lagrangian θ equation

$$\left(\alpha-\frac{\beta^2}{\gamma}\cos^2\theta+\sigma\gamma k^2\right)\ddot{\theta}+\left(\frac{\beta^2}{\gamma}\cos\theta\sin\theta+\sigma\gamma kk'\right)\dot{\theta}^2+D\sin\theta=0. \tag{10}$$

Also by substituting the expression for $\ddot{s}$ into the θ equation from the controlled Lagrangian gives

$$\left(\alpha-\frac{\beta^2}{\gamma}\cos^2\theta+\beta k\right)\ddot{\theta}+\left(\frac{\beta^2}{\gamma}\cos\theta\sin\theta+\beta k'\right)\dot{\theta}^2+D\sin\theta=0. \tag{11}$$

Comparing the two expressions gives $\sigma\gamma k^2 = -\beta k\cos\theta \Rightarrow k = \kappa\frac{\beta}{\gamma}\cos\theta$, where $\kappa = -\frac{1}{\sigma}$. Substituting for $\ddot{\theta}$ from the θ expression and k from the above expression we obtain the control law

$$u = \frac{\kappa\beta\sin\theta(\alpha\dot{\theta}^2 + \cos\theta D)}{\alpha - \frac{\beta^2}{\gamma}(1-\kappa)\cos^2\theta} \tag{12}$$

Finally, stabilization of the equilibrium $\theta = \dot{\theta} = \dot{s} = 0$ is stable if

$$\kappa > \frac{\alpha\gamma - \beta^2}{\beta^2} > 0. \tag{13}$$

4 Conclusion

Even though in this paper the controlled Lagrangian method was only demonstrated for the simple case of an inverted pendulum on a cart, there is a large class of mechanical systems to which it can be applied. I tried unsuccessfully to apply it to a particle in a magnetic field where the particle is controlled by varying the field strength. I also thought it would be interesting to try and apply it to the double pendulum on a cart.

References

[1] Bloch, A.M., N.E. Leonard and J.E. Marsden [2000] Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem. *IEEE Trans. on Systems and Control,* **45,** 2253-2270.

[2] Bloch, A.M., D.E. Chang, N.E. Leonard and J.E. Marsden [2001] Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping. *IEEE Trans. on Automatic Control,* **46,** 1556-1571.