CDS 205 report: Geometry of the Kaluza-Klein theory of Electromagnetism

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1 Introduction

During the 1920s Kaluza and Klein [1][2] put forward the concept of unifying the theories of general relativity and electromagnetism via the introduction of a fifth dimension. When viewed from the viewpoint of an observer living in space-time this extradimension transforms in the way required by the gauge transformations of electromagnetism. This idea was quickly reinterpreted into the mathematical language of connections living on principal bundles and generalised from the electromagnetic case to that of a general Yang-Mills field [3]. This reinterpretation marks a move from considering the higher dimension as a smooth manifold with a dimension curled up so that it can not be seen to a view point where a particle possesses an internal space which possesses certain group symmetries. It is this latter viewpoint which appears to have persisted.

Once the principal bundle formalism is established it is possible to extract equations of motion for a charged particle. These can be shown to be the projection of the geodesics in the bundle onto the base space. The deviation from geodesics in the flat space is interpreted as being due to a force acting on the particle.

Another way [4] of approaching the problem of generating the equations of motion of a particle in a Yang-Mills potential is to modify the symplectic structure of the cotangent bundle of the manifold by the addition of a two form based upon the connection of the principal bundle. This process corresponds to the process of minimal couplling in physics and leads to a geometric method of recovering the equations of motion. These different methodologies can be shown to be equivalent [6].

In this report I summarise some of the techniques used in the above approaches using the example of electromagnetism, which possesses a U(1) gauge symmetry, as a concrete example.

2 Kaluza-Klein Theory

The theory of Kaluza and Klein attempted to unify gravity and electromagnetism as due to the natural geometry of some special five dimensional manifold. Specifically they chose a manifold $R^4 \times S^1$. At the same time Utiyama [8] put forward a general view of forces as arising from gauge potentials based on symmetry groups. This theory can be regarded as equivalent to the mathematical theory of connections on principal bundles.

A principal bundle, P(M,G) is a fibre bundle over a base space M whose fibre F is identical with its strucure group G. A connection 1-form, ω on the tangent bundle TP is employed to effect a split of TP into vertical and horizontal subspaces,

$$T_pP = Hor \oplus Ver.$$

The vertical subspace is a subspace of T_pP which is tangent to the fibre G_p at the point u, $\pi(u) = p$. The horizontal subspace is the compliment of this vertical subspace. For practical purposes this split is accomplished using the Lie-algebra valued one-form, the connection ω. With this connection we say that a vector $X \in T_pP$ lies in the vertical subspace iff $< \omega, X >= 0$ where $< \cdot, \cdot >$ denotes the natural pairing between one-form and vector on T_pP.

To develop the Kaluza-Klein theory we seek to place a metric on this principal bundle[3]. The projection of geodesics on the bundle to the base space will then give the equations of motion of a charged particle moving in the Yang-Mills potential described by the structure group, G. This metric must obey several restrictions.

1. The metric must preserve the split between horizontal and vertical subspaces i.e. vertical and horizontal vectors must be orthogonal with respect to the metric
2. The metric on the horizontal subspace must be isomorphic to the metric on the base space.
3. The metric on the vertical subspace must be isomorphic to some metric on the Lie algebra of the structure group.

Let's work in the local trivialisation of the bundle and denote components on the fibre with indices a,b,c,etc. and components on the base by i,j,k,etc. Then the connection A has components A^a_j. Denoting coordinates in the bundle by greek letters which run over 1 to dim(M)+dim(G) we find that the metric takes the form

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij} + A^a_i A^b_j & A^b_j \\ A^a_i & g_{ab} \end{pmatrix}$$

The application of standard techniques to this metric allow the geodesic equations to be determined,

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + g^{im}g_{ab}Q^a F^b_{jm}\frac{dx^j}{ds} = 0,$$

where $Q^a = 2A^a_\alpha \frac{dx^\alpha}{ds}$. This latter quantity Q^a can be identified with the Yang-Mills equivalent of the charge divided by the mass of the particle. Notice that in flat space and for the EM field ($g_{ab} = 1$) this reduces to the usual Lorentz force on a particle.

For a definite example of this procedure let us consider the simple case of a charged particle moving on a flat background space through a magnetic field [5]. As mentioned previously the Kaluza-Klein configuration space for this particle is $Q = R^3 \times S^1$. The various conditions placed upon the components of the connection allow it to take the simple form $\omega_\alpha = (A_i, 1)$ we see that this generates a Kaluza-Klein metric

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij} + A_i A_j & A_j \\ A_i & 1 \end{pmatrix}.$$

From this metric we form a modified kinetic energy lagrangian.

$$\begin{aligned} L(x, \dot{x}, \theta, \dot{\theta}) &= \tfrac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta \\ &\tfrac{1}{2} m \|\dot{x}\|^2 + \tfrac{1}{2}(A \cdot \dot{x} + \dot{\theta})^2 \\ &\tfrac{1}{2} m \|\dot{x}\|^2 + \tfrac{1}{2} \| < \omega, (x, \dot{x}, \theta, \dot{\theta}) > \|^2 \end{aligned}$$

where $q = (x, \theta) \in Q$, $x \in R^3$ and $\theta \in S^1$. The Euler-Lagrange equations for this lagrangian will recover geodesics of the above metric. Inspection reveals that θ is a cyclic variable and so

$$\frac{d}{dt} p_\theta = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0$$

leading to $p_\theta = A \cdot \dot{q} + \dot{\theta}$ being a constant of the motion. We identify this conserved quantitiy with the charge of the particle.

The equations of motion for the spatial part of the problem yield

$$\frac{dp_i}{dt} = \frac{d}{dt}(m\dot{x}_i + p_\theta A_i) = -\frac{\partial L}{\partial x^i}$$

which give us the usual Lorentz force equations.

This approach has very general application, and provides a very intuitive picture of the gauge theory. One interesting point is that paths which follow closed loops in the base space do not necessarily close when lifted into the bundle. Consider a closed curve c(t) on the base space. This is the projection of some curve on the bundle $\tilde{c}(t)$. It is quite possible that the end points $\tilde{c}(0)$ and $\tilde{c}(1)$ differ by some vertical vector while still having the same projection onto the base space. This vertical vector is related to the field strength of the Yang-Mills field also known as the curvature of the connection.

3 Minimal Coupling

The standard proceedure in physics for generating the equations of motion of a particle in an electromagnetic field is that of the minimal coupling scheme by setting $p \to p - eA$ in the Hamiltonian. It was observed that this proceedure was equivalent to taking the usual Hamiltonian, but altering the symplectic structure on T^*M by the addition of a two form dA to the standard symplectic form. The equations of motion with respect to this Hamiltonian and the modified symplectic structure are the desired ones for a particle moving in an em field. This procedure generalises to the case of the Yang-Mills field where once again we formulate our theory on a principal bundle. Following Sternburg [4] we consider a principal G-bundle $P \to X$ with G a Lie Group. Let X be a symplectic manifold with symplectic form ω We next take a hamiltonian G-space F with a symplectic form Ω. The group G acts on F as a group of symplectic diffeomorphisms. From these two bundles we form the associated bundle $P \times_{/G} F$. The connection on P then determines a symplectic structure on $P \times_{/G} F$. This associated bundle has a unique, closed form Ω_A defined by $d < A, \Phi > + \Omega = \pi^* \Omega_A$. We use this to form the modified symplectic structure on $P \times_{/G} F$ by forming the closed two-form $\omega + \Omega_A$. The equations of motion are then Hamilton's equations with respect to this symplectic structure.

For the case of the electromagnetic field we have G=U(1) and F=e. For a connection 1-form A we thus obtain $\Omega_A = d < A, e >$ from which we recover the modified symplectic form $\omega + edA$ where we identify e as the electric charge. This leads us the equations of motion for a charged particle in a magnetic field $B = \nabla \times A$.

4 Equivalences

The two methods outlined above appear at first glance to be very different. In the Kaluza-Klein approach we put a metric on the principal bundle P and found our equations of motion from the geodesic equations. In the minimal addition scheme we modified the symplectic structure of the associated bundle $P \times_{/G} F$ and found our equations of motion from Hamilton's equations with this structure. These two methods are in fact equivalent in a natural way as was shown by Montgomery [6].

In fact Montgomery lists a series of methodologies for calculating the equations of motion for a particle in a Yang-Mills force [7]. These methods are all equivalent and represent different ways of looking at the same problem. The net result is that there are many ways to describe motion on the bundle all of which have the same projection onto the base space. These include forming a purely horizontal hamiltonian, the Kaluza-Klein approach and the the modified symplectic geometry appraoch.

5 Conclusions

The original notion of Kaluza-Klein to exploit extra dimensions in an attempt to unify theories of gravity and other forces showed initial promise. The original formulation has been superceded by the use of connections on principal bundles to describe gauge theories. This has great application in a variety of circumstances. Beyond application to fundamental physics the ideas of gauge theories can be applied to problems as diverse as the falling cat problem, the motion of ameoba and a myriad of control problems.

The general approach in all of these problems is to formulate the theory on a configuration space Q which posesses some symmetry group G. The "shape space" Q/G which forms the base space of the bundle then takes the role of the space of either the control space or the observable space coordinates. A connection living on T^*P is then used to effect a split between horizontal and vertical subspaces. Different, but equivalent, methodologies are available for extracting the equations of motion from the bundle and connection. Some of these apply only in certain situations, e.g. Kaluza-Klein for which the Lie-Group must possess a bi-invariant metric in order to form the Kaluza-Klein metric.

Despite the great effectiveness of the mathematical structure provided by connections on principal bundle it seems that the original principle of Kaluza-Klein was lost along the way. Generating higher dimensional riemannian manifolds which allow the reproduction of the observed standard model group symmetries in a natural way seems like a sensible plan. It appears that the complexity of determining appropriate topologies on the space has dissauded progress on this avenue except perhaps in the arena of string theory.

References

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