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SID 114 39 181

MATH 189 SP 95
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ON THE ROLE OF TORI IN MECHANICS

"A Lemma is essentially a Theorem, except that it is harder to prove."

D. SARASON

I / Introduction

I initially wanted to study a particular integrable system. Upon your suggestion, I attempted to read the article on the Kowalewski Top. However, I soon realized that its contents were too complicated and that any paper on the subject would have been "brainless regurgitation".

Thereupon I consulted Prof. Reshetikhin who suggested that I study a series of articles on Integrable maps. A closer survey of these articles revealed to me that I was not familiar with the necessary notions of Algebraic Geometry and that the few theorems that I could claim to understand would not unite to a paper.

While studying these articles however, I realized that at the foundations of Integrability Theory was the so-called "Liouville's Theorem" (one of the many so-called), that I neither knew, nor had any idea how to prove. The proof of the entire theorem is spelled out in Arnold's "Mathematical Methods of Classical Mechanics", leaving many of the details as exercises for the reader. These I have tried to prove as best possible and they are to be found in the paper as "Propositions". Due to lack of time, I have not been able to study the entire proof, so that I present the first half (more or less).

II / Statement of Theorem / Partial Proof / Corollaries

We begin the main part of the paper with a:

definition: Two functions F_1, F_2 on a symplectic manifold are IN INVOLUTION if their Poisson Bracket is identically equal to zero.

Next we state Liouville's Theorem on Integrability:

Theorem: (Liouville) let M^{2n} be a symplectic manifold. Suppose that we have n functions on M^{2n} that are in involution: $F_1, \ldots, F_n$ such that $(F_i, F_j) \equiv 0 \ \forall i,j \quad i,j: 1, \ldots, n$.
Consider a levelset of these functions:

$M_f = \{ x \in M^{2n} : F_i(x) = f_i, \ i = 1, \ldots, n \}$

Assume that the functions are functionally independent on M_f. Equivalently, their differentials are linearly independent at each point of M_f. Then we have the following:

1) M_f is a smooth manifold, invariant under the phase flow with Hamiltonian function F_1.

2) If the manifold is compact & connected, then it is diffeomorphic to the n-dimensional torus $T^n = \{ (\varphi_1, \ldots, \varphi_n) (\mathrm{mod}\ 2\pi) \}$.

3) The phase flow with the above Hamiltonian function determines conditionally periodic motion on M_f.

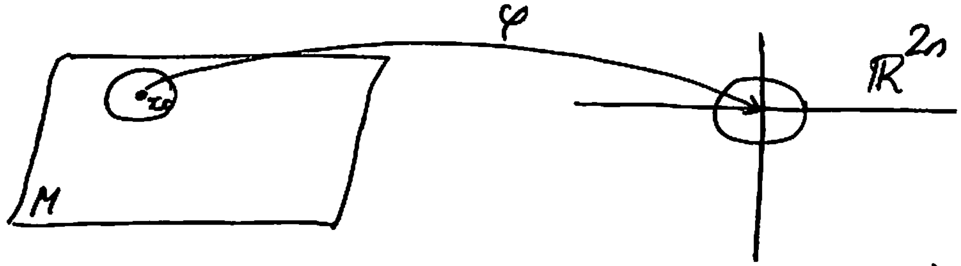
4) The canonical equations of F_1 can be integrated by quadratures.

Let us consider the level set as described in the Theorem:

$$M_f = \{x : F_i = f_i = (\text{constants}), i = 1, \dots, n\}$$

Proposition1: M_f is an n-dimensional submanifold of the $2n$-dimensional phase space.

proof: By hypothesis, the n differentials (1-forms) dF_i are linearly independent. Consider a point x_0 on M_f and a neighborhood of this point that is inside a chart about the origin in $\mathbb{R}^{2n}$



next consider the functions $f_i = \varphi \circ F_i(x)$. For simplicity in other words, consider a small sphere about the origin s.t. we can expand the functions f_i which represent the F_i about the origin in a Taylor series and neglect the second & higher order terms. Thus we have:

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x_\alpha}(0)\, x_\alpha + O(x_\alpha^2)$$

where x_α are the coordinates on $\mathbb{R}^{2n}$. Then, neglecting the second order terms we have the following system of linear equations: $\displaystyle\sum_{\alpha=1}^{2n}\sum_{i=1}^{n} \frac{\partial f_i(0)}{\partial x_\alpha} = 0 \implies$

$$\begin{cases} \dfrac{\partial f_1(0)}{\partial x_1} + \dfrac{\partial f_2(0)}{\partial x_1} + \dots + \dfrac{\partial f_n(0)}{\partial x_1} = 0 \\ \vdots \\ \dfrac{\partial f_1(0)}{\partial x_{2n}} + \dots\dots + \dfrac{\partial f_n(0)}{\partial x_{2n}} = 0 \end{cases}$$

but this is a system of n equations and $2n$ variables thus we have an n-dimensional solution space (since we have n free parameters). This completes the proof. M_f is an n-dimensional submanifold since every point x of M_f has such a neighborhood.

<u>Lemma 1</u>: On the n-dimensional manifold M_f there exist n tangent vector fields which commute, and which are linearly independent at every point.

proof: The proof relies on the following:

<u>Proposition 2</u>: If we have a symplectic manifold (M^{2n}, ω^2) ~~[illegible]~~ then there exists a 1-1 correspondence of 1-forms onto ~~vector~~ tangent vectors of this space.

proof: let TM_x be the tangent space at x. let $\{e_i\}$ form a basis for TM_x. Then we define a homomorphism from TM_x to T^*M_x by:

$$\varphi: TM_x \to T^*M_x \; : \; \varphi(e_i) = \underbrace{\omega^1_{e_i}}_{(1\text{-form})}(v) = \omega^2(e_i, v)$$

where $v \in T_xM$.

As we know from linear algebra, the linear map is completed determined by its effect on the basis elements so that here the map is determined. Let us prove injectivity. Suppose that $\omega^1_{e_1} = \omega^1_{e_2} \Longleftrightarrow$

$$\omega^2(e_1, v) = \omega^2(e_2, v) \quad \forall v \in T_xM$$

$$\Longleftrightarrow \quad \omega^2(e_1 - e_2, v) = 0$$

but ω^2 is a closed nondegenerate 2-form thus $e_1 = e_2$ as desired. We have an injective map of finite dimensional vectorspace. This implies that they are isomorphic.

Thus we have n vector fields which we denote X_{F_i}. Since φ is an isomorphism, the X_{F_i} are linearly independent. Moreover, the fields commute since the Poisson Brackets of their corresponding differentials vanish. This further implies that the derivative of any F_i in the direction of any ~~#~~ X_{F_j} vanishes (since $\mathcal{L}_{X_{F_j}}(F_i) := dF_i \cdot X_{F_j}$).
This proves the Lemma.

Lemma 1 shows that the n commuting flows φ_i of the vector fields X_{F_i} leave the manifold invariant.

We now turn to Lemmas 2 and 3 which are topological statements that underlie the proof of the Theorem.

<u>Lemma 2:</u> Let M^n be a compact connected differentiable n-dimensional manifold, on which we leave n pairwise commuting vector fields that are linearly independent at each point. Then M^n is diffeomorphic to the n-torus T^n.

<u>proof</u>: let φ_i^t be the flows of the n commuting, linearly independent vector fields. Since the vector fields commute, so do the flows φ_i^t. We thus define a group action of the commutative group under addition $\mathbb{R}^n = \{\vec{t}\}$: $\varphi^{\vec{t}}: M \longrightarrow M$ such that $\varphi^{\vec{t}} = \varphi_1^{t_1} \cdots \varphi_n^{t_n}$.
Then we have: $\varphi^{\vec{t_1}+\vec{t_2}} = \varphi^{\vec{t_1}}\varphi^{\vec{t_2}}$ for $\vec{t_1}, \vec{t_2} \in \mathbb{R}^n$ as required for a group action on a set.

If we fix some point $x_0 \in M$, then $\varphi(\vec{t}) = \varphi^{\vec{t}} x_0$ and the point moves along each of the flows for the corresponding times t^i.

PROPOSITION 3: The map φ maps a neighborhood V of the origin in $\mathbb{R}^n$ diffeomorphically onto a neighborhood U of x_0 ($x_0 \in U \subset M$), provided V is sufficiently small.

proof: We choose V so small that we can neglect the second and higher order terms in the Taylor expansion of φ about the origin:

$$\varphi(\vec{t}) = \varphi(0) + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(0)\, x_i + O(x_i^2 + x_i x_j \cdots)$$

we obtain an $n \times n$ matrix:

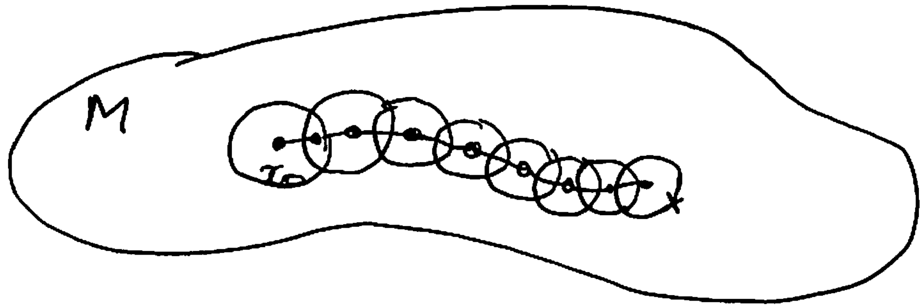
$$\begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial \varphi_n}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_2} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{bmatrix}$$

but this matrix is invertible since the vector fields are linearly independent.

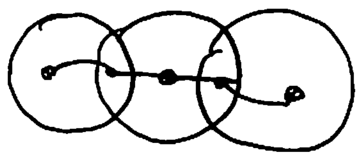
Therefore, the map is 1-1 as desired.

PROPOSITION 4: The map φ is onto M.

proof: It will be sufficient to show that given a point x_0 we can reach any other $x \in M$. Let us cover M by neighborhoods of the type described in the previous proposition. Then a finite subset of these neighborhoods satisfies the condition that one contains x, another x_0 and the ones in between overlap so that the set is connected, because M is a compact manifold. The picture is then:



Then in each of the overlapping neighborhoods, call them N_i we choose points:



in such a manner that for each N_i, we have two points n_{i1}, n_{i2} s.t. $n_{i1} \in N_{i-1} \cap N_i$, $n_{i2} \in N_i \cap N_{i+1}$.

Since $\mathbb{R}^n$ as defined acts as a group on M then for every pair of points (n_{i1}, n_{center}), (n_{i2}, n_{center}) there exist $t_{n_{i1}}, t_{n_{i2}}$ that takes $n_{i1} \to n_{center}$ and $n_{center} \to n_{i2}$ respectively.

In this manner we connect x_0 with x via chain links determined by the $\{t_{n_i}\}$ and φ is onto.

We now approach the 3rd Lemma which will allow us to prove Lemma 2. For this, we first need a:

<u>definition</u>: The stationary group of a point x_0 is the set Γ of points $\{t\} \in \mathbb{R}^n$ such that $\varphi^t x_0 = x_0$

<u>PROPOSITION 5</u>: Γ, as defined above is indeed a group. Moreover, it is independent of the point in M, that it acts on.

<u>proof</u>: 1) the identity certainly belongs to Γ since $\varphi^0 x = x$.

2) let $s, t \in \Gamma$ we must show that $s+t \in \Gamma$:

$$\varphi^{s+t}(x_0) = \varphi^s \varphi^t (x_0) = \varphi^s (x_0) = x_0 ;$$

3) finally, every element clearly has an inverse, namely $-t$ for t for example.

thus Γ at x_0 is a subgroup. Moreover:
suppose that ~~[illegible]~~ $x = \varphi^r(x_0)$, let $t \in \Gamma$
then $\varphi^t x = \varphi^{t+r}(x_0) = \varphi^r \varphi^t x_0 = \varphi^r(x_0) = x$
since $\mathbb{R}^n$ is commutative. This proves the proposition.

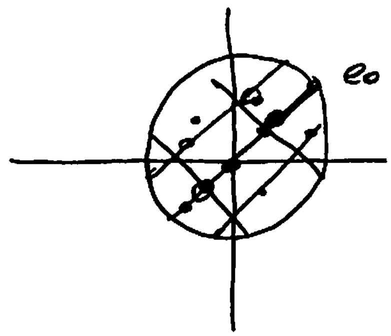
<u>PROPOSITION 6</u>: The group Γ is a discrete subgroup, that is, the points of Γ are isolated in $\mathbb{R}^n$.

<u>proof</u>: by PROPOSITION 3, there exists a sufficiently small neighborhood V of the point $0 \in \mathbb{R}^n$ such that φ is 1-1. The point 0 satisfies $\varphi^0(x) = x$. Such a neighborhood exists for every point $t \in \Gamma$ since if $t \in \Gamma$ then the map from V to $t+V$ defined by $f(x) = t + x$ is certainly 1-1 and so the composite is 1-1 in $t+V$, which proves the proposition.

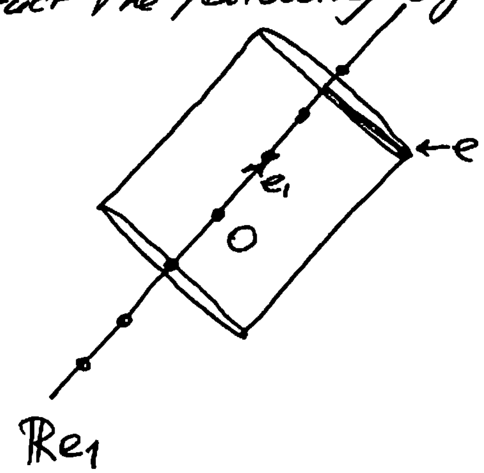
If we let $e_1, \dots, e_k$ be linearly independent vectors in $\mathbb{R}^n$, then their linear combinations $\sum m_i e_i$ where $m_i \in \mathbb{Z}$ certainly form a subgroup because we can always add elements of Γ (we assume here that $e_i \in \Gamma$) and add each element to itself m_i times as Γ is a group, hence the above claim follows. It turns out that the discrete subgroups in $\mathbb{R}^n$ are all of this type for appropriately chosen $\{e_i\}$. This is the content of:

<u>Lemma 3</u>: Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exist k $(0 \leq k \leq n)$ linearly independent vectors $e_1, \dots, e_k \in \Gamma$ such that Γ is exactly the set of their integral linear combinations.

proof: Consider $\mathbb{R}^n$ with the usual inner product. We will prove the lemma "by induction" on the number of distinct e_i's as follows:
certainly Γ always contains 0. If $\Gamma = \{0\}$ then we are done. Suppose $e_0 \in \Gamma$, $e_0 \neq 0$. Then consider the line $\mathbb{R}e_0$. Consider moreover a disk of radius $|e_0|$:



This is a compact subset of $\mathbb{R}^n$ and we know by PROPOSITION 3 that it can contain only a finite number of points of Γ. In particular, only a finite number of points of Γ on $\mathbb{R}e_0$. Then a closest point exists. Let this point be e_1. Its integral multiples are precisely the points satisfying $\mathbb{R}e_1 \cap \Gamma$. Moreover, there can be no points in between some me_1 and $(m+1)e_1$ since if $m < \lambda < (m+1)$ and $\lambda e_1 \in \Gamma$ then $\lambda e_1 - me_1 \in \Gamma$ and this point is closer to 0 than e_1, a contradiction.
If there are no points of Γ outside those of me_1 then we are done. If not, then let e be a point off $\mathbb{R}e_1$ and construct the following cylinder:



The argument parallels that for the construction of e_1, namely, in this cylinder there is a finite number of points $\in \Gamma$ and we can choose one whose distance from $\mathbb{R}e_1$ is the smallest. Then we let this point be e_2 and observe that outside the cylinder, there cannot be a point closer to $\mathbb{R}e_1$ than e_2 since if we shift it down into the cylinder, we get a point closer to $\mathbb{R}e_1$, a contradiction. Thus, Γ now has all points of the form $m_1 e_1 + m_2 e_2$ where $m_1, m_2 \in \mathbb{Z}$. If this exhausts Γ then we are down. Otherwise we continue in higher dimensions as we have been working so far: we consider a point not lying in the plane (since no other points can be in the plane by virtue of our "integral shifting" procedure) etc. In general, suppose that we have $e_1, \ldots, e_\ell$ and $e \notin \sum_{i=1}^{\ell} m_i e_i$. Then, in $\mathbb{R}^n$, we can project e onto the e_i and construct an $(\ell+1)$ dimensional equivalent of our cylinder with finite number of points $\in \Gamma$ not in $\{\sum_{i=1}^{\ell} m_i e_i\}$, then choose among this finite set the closest one and by virtue of shifting show that only integral linear combinations of now $e_1, \ldots e_\ell, e_{\ell+1}$ can exists. Thus we either reach $\underline{n=k}$ or Γ is exhausted before. This completes the proof of Lemma 3.

We are now in the position to prove Lemma 2. We have but one more proposition to prove, after the following definitions:

let $T^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \ldots \varphi_k, x_1, \ldots, x_{n-k})\}$ $\varphi_i \pmod{2\pi}$.

Consider the natural map:

$$p : \mathbb{R}^n \longrightarrow T^k \times \mathbb{R}^{n-k}$$

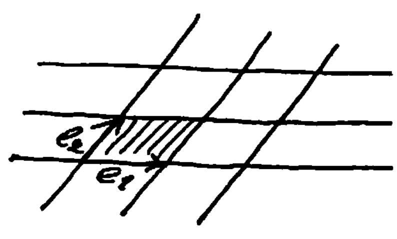
$$p(\varphi_1, \ldots, \varphi_k, x_1, \ldots, x_{n-k}) = (\varphi_i \pmod{2\pi}, x_i)$$

let the points f_i have coordinates $(\varphi_i = 2\pi, \varphi_j = x_j = 0)$.
These points f_i are mapped to zero under ρ.
Moreover, let $e_1, \dots, e_k \in \Gamma \subset \mathbb{R}^n$ be the generators of Γ as described in Lemma 3. We define an operator A:
$A: \mathbb{R}^n = (\varphi_i, x_j) \longrightarrow \mathbb{R}^n = \{\vec{t}\}$ as in Lemma 2, such that $f_i \longmapsto e_i$. Such a map is always possible since linear transformations are determined by mapping n linearly independent quantities to any ~~a~~ n quantities, in particular $\{e_i\}$ which <u>are</u> linearly independent. We recall that $\mathbb{R}^n = \{\vec{t}\}$ gives charts on M_f and $\mathbb{R}^n$ gives charts on $T^k \times \mathbb{R}^{n-k}$.

<u>PROPOSITION 7:</u> The following diagram commutes for a given operator $\tilde{A}$ which is a diffeomorphism $\tilde{A}: T^k \times \mathbb{R}^{n-k} \longrightarrow M_f$:

$$\begin{tikzcd}
\mathbb{R}^n = \{(\varphi_i, x_i)\} \arrow[r, "A"] \arrow[d, "p"'] & \mathbb{R}^n = \{\vec{t}\} \arrow[d, "\varphi"] \\
T^k \times \mathbb{R}^{n-k} \arrow[r, "\tilde{A}"] & M_f
\end{tikzcd}$$

<u>proof</u>: the intuition is as follows: the $\{e_i\}$ define k-dimensional parallepipeds with the property that at the vertices the flow is trivial under φ but inside $\varphi^t x \neq x$. The two-dimensional picture is:



A point initially in $\mathbb{R}^n$ is mapped to the set $\{\vec{t}\}$. This is a "change of basis". φ "collapses" these ~~parallepipeds~~ points to parallepipeds with "trivial flow" vertices. On the other hand p collapses points in $\mathbb{R}^n$ onto T^k tori. We now wish to transform the tori to the corresponding parallepipeds.
(Remark: I fear that this intuition, though it seems elegant to me, is probably false on more rigorous topological grounds.)

Let us look at the map more precisely:
Consider (φ_i, x_i) in $\mathbb{R}^n$. Let $A(\varphi_i, x_i) = (b_i, c_i)$ in $\{\vec{E}\} = \mathbb{R}^n$.
These can now be taken "modulo $\{e_i\}$". To see this let

$$m_i \in \mathbb{Z} \text{ satisfy: } \begin{cases} m_i < b_i \\ m_j < c_j \end{cases}$$

then $(b_i, c_j) = (m_i, m_j) + (r_i, q_j)$ such that

$$\begin{cases} 0 \leq r_i < |e_i| \\ 0 \leq q_j < |e_j| \end{cases}$$

thus
$$\varphi^{(b_i, c_j)}(x) = \varphi^{(m_i, m_j) + (r_i, q_j)}(x)$$
$$= \varphi^{(r_i, q_j)} \circ \varphi^{(m_i, m_j)}(x)$$
$$= \varphi^{(r_i, q_j)}(x).$$

On the other hand we have, via p: $(\varphi_i, 0) \pmod{2\pi}$.
We now need $\tilde{A}(\varphi_i, 0)(x) = \varphi^{(r_i, q_j)}(x)$
now define the function $h(\varphi_i, 0)$ in the following manner:

$$\begin{cases} h_i(\varphi_i) = \dfrac{\varphi_i}{2\pi} |e_i| & i = 1, \dots, k \\ h_j(0) = q_j = j\text{th component of } A(\varphi_i, x_j) \end{cases}$$

then the function h is certainly ~~differentiable~~ differentiable and (assuming $h_j(0)$ definition is allowed as done) then at least the h_i's are 1-1. I must admit that I do not see how to construct the h_j so that they are 1-1. Assuming, in resignation, that this is done, then let $\tilde{A} = g \circ h$, the composite of 2 diffeomorphisms, a diffeomorphism, and the diagram commutes. This "proves" Lemma 2 since M_f is compact by hypothesis, moreover $k = n$.

Thus M_f is an n-dimensional torus. (we observe that for $k=n$ the above problem doesnot arise).

This proves the first two assertations of Liouville's Theorem.
We note two immediate corollaries:

<u>COROLLARY 1</u>: Any 1-dimensional system is integrable.
proof: the Hamiltonian is in involution with itself.

<u>COROLLARY 2</u>: If a 2-dimensional system has a "cyclic" coordinate, it is integrable.
proof: $(F_1, H) = 0$ "by cyclic coordinate."

More generally, if we have n-conserved quantities (i.e. constant on Manifold) that correspond to n linearly independent functions, then the system is integrable. An example is the symmetric heavy Lagrange top.

We now give an example (explicit, that is) of a 1-dimensional system. Generally, let $H = \text{Energy} = \frac{1}{2} m\dot{q}^2 + V(q)$ this can be written as: $(\dot{q})^2 = \frac{(H - V(q))\,2}{m}$ or $\frac{dq}{dt} = \sqrt{H - V(q)}$ (absorbing constants)

$$\text{or,} \quad dt = \frac{dq}{\sqrt{E - V(q)}} \Longrightarrow t = \int \frac{dq}{\sqrt{E - V(q)}}.$$

Where we assume that the energy is constant..., as in the following example:

$\boxed{\text{II}}$ $V(q) = d/q^2$

1) $d > 0$
2) $d < 0$

1), 2)
$$\int \frac{dq}{\sqrt{a \pm |d|/q^2}} = \int \frac{q\,dq}{\sqrt{aq^2 \pm |d|}} = \frac{1}{\sqrt{a}} \int \frac{q\,dq}{\sqrt{q^2 \pm |d|/a}}$$

but this is a "standard integral".

Case 1)

$$\frac{1}{\sqrt{a}} \int \frac{q\,dq}{\sqrt{q^2 + d/a}} = \frac{1}{\sqrt{a}} \sqrt{q^2 + d/a}$$

then $t = \frac{1}{\sqrt{a}} \sqrt{q^2 + d/a} \implies \sqrt{a}\, t = \sqrt{q^2 + d/a}$

$$\implies a t^2 = q^2 + d/a$$

$$\implies q^2 = a t^2 - d/a$$

$$\boxed{q(t) = \pm \sqrt{a t^2 - d/a}}$$

Case 2)

$$\frac{1}{\sqrt{a}} \int \frac{q\,dq}{\sqrt{q^2 - d/a}} = \frac{1}{\sqrt{a}} \sqrt{q^2 - d/a}$$

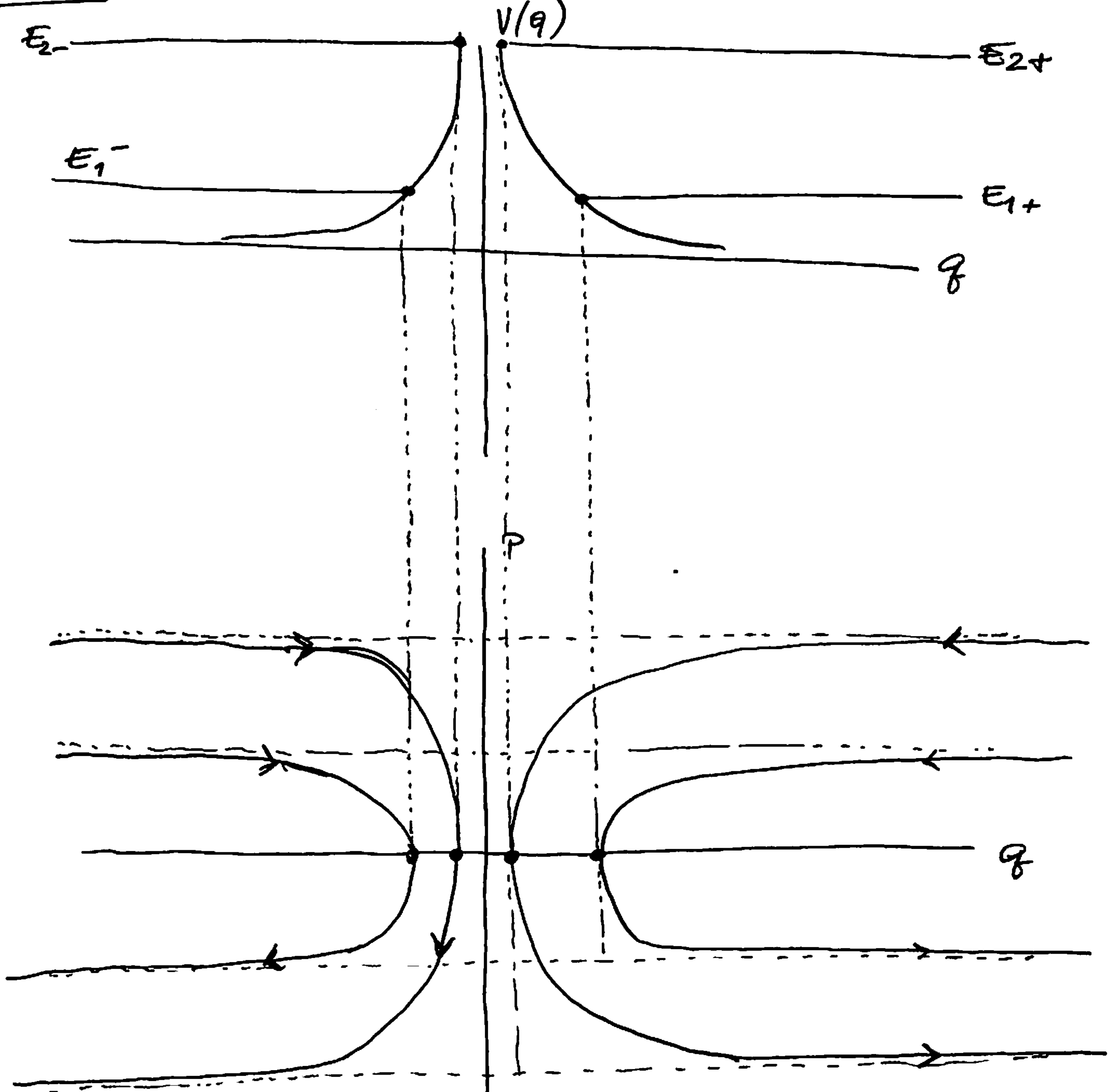
then, proceeding as above:

$$\boxed{q(t) = \pm \sqrt{a t^2 + d/a}}$$

this describes the trajectories q as functions of time completely.

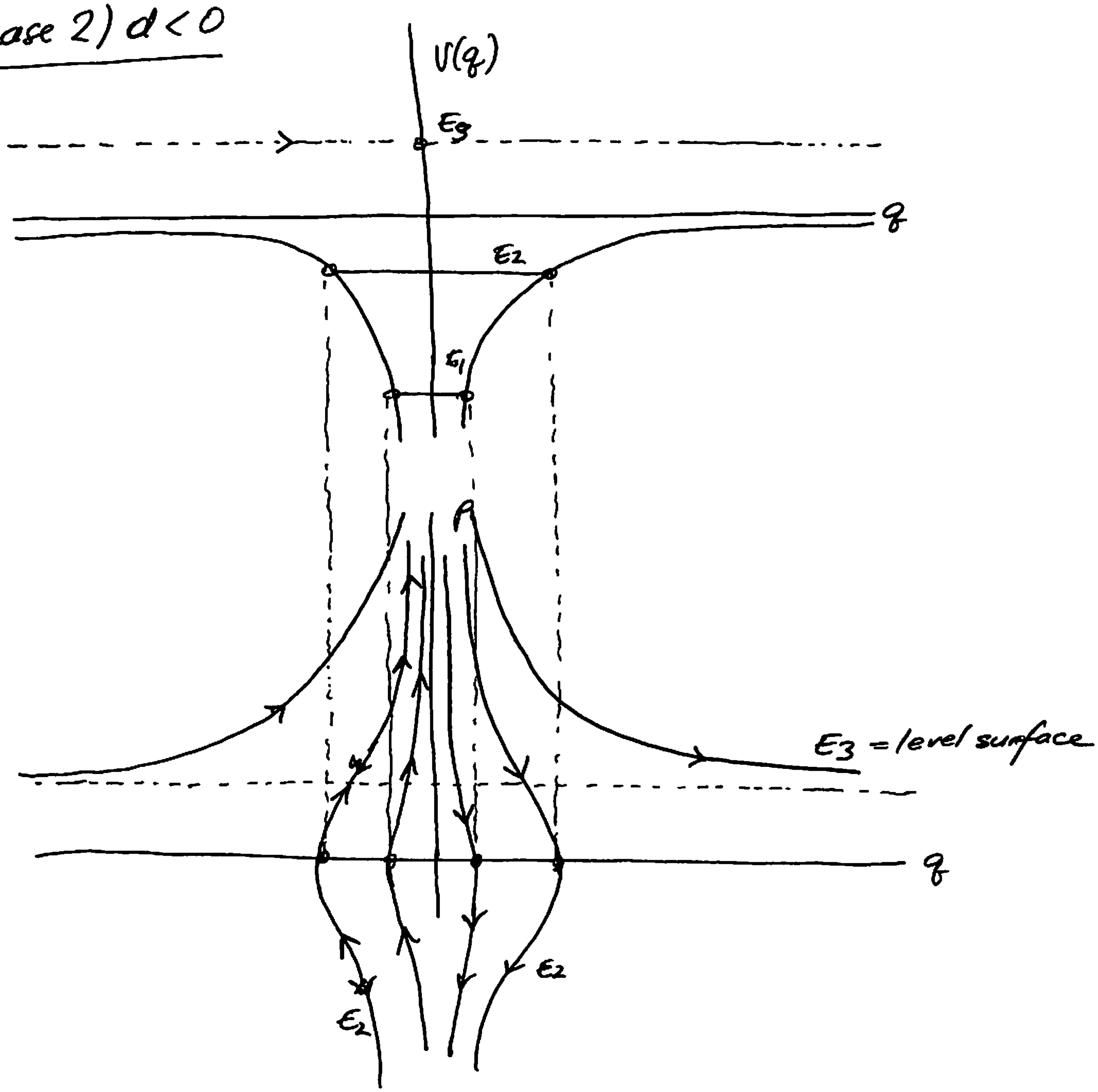
$\boxed{\text{II}}$ trajectories & level surfaces in phase space

case 1: $d > 0$



"as q decreases, the potential increases and therefore, the kinetic energy must decrease also since the total energy on each level surface is constant."

case 2) $d < 0$



III / CONCLUSION

As the reader will have observed, this paper is in many ways incomplete and therefore unsatisfactory. Unfortunately, I spent too much time "toying" with the articles on integrable maps & on the Kowalewski Top before I realised that I could not write a decent paper on any one of these topics – that I would have to spend much more time on these works, which I hope to do this summer. Certainly, the proof of integrability is the highlight of this theorem, but time has made it elude me. However, I am happy that I wrote my paper on this theorem because it lies at the base of the theory of Integration and one should not proceed to study the more advanced topics of a subject without knowing some of the foundations, a reality that this paper has brought me closer to.

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