rough draft

The KdV Equation: A CDS 241a Project

by Mason A. Porter

3/17/98

1 Introduction

1.1 History

In recent years, soliton technology has become a piece of popular science. Its propoonents have been churning out vast quantities of information in their latest efforts to impress the scientific community and the world at large. However, the idea of a solitary wave (a "soliton") has been present since the 19th century. John Scott Russell, a Scottish engineer, sought to create a more efficient hull design for canal boats. In August 1834, he stood beside Union Canal near Edinburgh to observe the movement of a boat being pulled by a pair of horses. When the rope pulling the boat snapped, the boat stopped moving and its prow dropped down. Russell saw a smooth solitary wave emerge from the water, gather around the prow, and continue along its way down the channel. The wave held its shape as it travelled, although it decreased somewhat in height. Nevertheless, it was considered "common knowledge" that waves could not behave in this manner, so Russell's discovery was all but ignored.

Russell was eventually proven correct. Proof that solitary waves were theoretically possible came from Boussinesq's equation in 1872 and the Korteweg-de Vries equation in 1895. Some results were obtained, but the field lay all but barren for another 70 years. In 1965, Martin Kruskal and Norman Zabusky studied the KdV equation numerically and determined that nonlinear solitons could occure naturally. They also obtained another striking result. Although the equation is nonlinear and it was expected that two solitary waves travelling at different speeds would interact with each other in a complicated fashion, Kruskal and Zabusky found that the interaction between the waves was only temporary and that the waves quickly recovered their original shapes and velocities. In fact, the name "soliton" arose from the fact that these elastic collisions resembled those of colliding elementary particles. After this point, the theory advanced exponentially.

1.2 Motivation

A common choice of configuration space for classical field theory is an infinite-dimensional vector space of functions or tensor fields on space or spacetime. The elements of such a vector space are known as **fields**. As in the finite-dimensional case, one can derive Hamilton's equations for infinite-dimensional systems. This leads to many familar partial differential equations, including the Korteweg-de Vries (KdV) Equation. (For example, see chapter 3 of *Introduction to Mechanics and Symmetry* by Jerrold Marsden and Tudor Ratiu [2].) The KdV equation can be analyzed in other—more classical—methods as well. Such an analysis can be found in *Linear and Nonlinear Waves* by Gerald Whitham [3].

2 The Variational Approach

Let Z denote the vector subspace $\mathcal{F}(\mathbb{R})$ consisting of those functions u whose absolute value decreases sufficiently fast as x tends to positive and negative infinity so that the formulas that appear in this paper are valid. This assumption, moreover, simplifies the process of integration by parts considerably (because it forces the boundary terms to vanish), and it thus justifies computations that would otherwise be formal.

We will begin with a symplectic structure that is somewhat more complicated than necessary. Pair Z with itself using the L^2 inner product. Additionally, let the KdV symplectic structure Ω be defined by equation [1]. In this equation, $\hat{u}$ denotes the integral of u from $-\infty$ to x. This is also known as the primitive of u. The form Ω is clearly skew-symmetric. Note that if $u1 = \frac{\partial v}{\partial x}$ for some $v \in Z$, then a few lines of calculation verify equation [2]. This shows that Ω can be written in the form of equation [3].

We now need to prove weak nondegeneracy of Ω. To do this, we check that if $v \neq 0$, there exists a $w \in Z$ such that $\Omega(w, v)$ does not vanish. Indeed, for $v \neq 0$, we let $w = \frac{\partial v}{\partial x}$. The vector w does not vanish because $v(x) \longrightarrow 0$ as $\|x\| \longrightarrow \infty$. Hence, by equation [3] leads to equation [4].

Suppose that a Hamiltonian $H : Z \longrightarrow \mathbb{R}$ is given. We claim that the corresponding Hamiltonian vector field X_H is given by equation [5]. Plug equation [3] into equation [6]. Using equation [5], we see that the corresponding Hamilton equations take the form of equation [7], where the subscripts in equation [7] and in the ones immediately following it denote derivatives. As a special case, consider the Hamiltonian H_1 given in equation [8]. This leads to the **one-dimensional transport equation** ($u_t + uu_x = 0$, equation [9]). One can analyze this equation using the method of characteristics. As another example, consider the Hamiltonian H_2 defined by equation [10]. This leads to the KdV equation ($u_t + 6uu_x + u_{xxx} = 0$, equation [11]), which describes shallow water waves. The above are one two examples of a famous complete set of conserved integral quantities. There are many methods to derive them, including the use of generating functions.

One can analyze these conserved quantities by defining a Poisson bracket called the **KdV Bracket**. Using the definition of the bracket ($\{F, G\}(z) = \Omega(X_F(z), X_G(z))$, equation [12]), the symplectic structure, and the Hamiltonian vector field discussed above, one finds that the KdV Bracket $\{F, G\}$ satisfies equation [13]. One can use this to show that $\{F_i, F_j\} = 0$, where $i, j \in$

$\{0,1,2,3\}$, the Poisson bracket is the KdV bracket, and where the F_i are defined in equations [14]—[17]. Recall that $F_2(u)$ is the KdV Hamiltonian.

2.1 Travelling Waves

Consider travelling wave solutions of the KdV equation. In other words, let $u = \phi(x-ct)$, where $c > 0$ is a constant and ϕ is a positive function. By direct substitution, it is easily shown that u satisfies the KdV equation if and only if ϕ satisfies $c\phi'' - 6\phi\phi' - \phi''' = 0$ (equation [18]). Integration shows that $c\phi' - 3\phi^2 - \phi'' = C(equation[19])$, where C is a constant that will be determined later. This equation is Hamiltonian in the canonical variables (ϕ, ϕ') with Hamiltonian function given by $h(\phi, \phi'_= \frac{1}{2}(\phi')^2 - \frac{c}{2}\phi^2 + \phi^3 + C\phi$ (equation [20]). By conservation of energy, $h(\phi, \phi') = D$. Therefore, ϕ' is expressed as in equation [21]. Writing $s = x - ct$ gives us equation [22].

We seek solutions that vanish at $\pm\infty$ and have derivatives that vanish there as well. It then follows that $C = D = 0$, which gives us equation [23]. In this equation, K is a constant of integration. For $C = D = 0$, the Hamiltonian becomes $h(\phi, \phi') = \frac{1}{2}(\phi')^2 - \frac{c}{2}\phi^2 + \phi^3$ (equation 24). Our system then has equilibria given by $\frac{\partial h}{\partial \phi} = 0, \frac{\partial h}{\partial \phi'} = 0$ of $(0,0)$ and $(\frac{c}{3}, 0)$. The matrix of the linearized Hamiltonian system is then given by formula [25]. This shows that $(0,0)$ is a saddle point and $(\frac{c}{3}, 0)$ is spectrally stable.

Using the second variation criterion on the potential energy $-\frac{c}{2}\phi^2 + \phi^3$ at $(\frac{c}{3}, 0)$ shows that this equilibrium is stable. Hence, if $(\phi(s), \phi'(s))$ is a homoclinic orbit beginning and ending at $(0,0)$, the value of the Hamiltonian on it is $H(0,0) = 0$. From equation [24], it follows that $(\frac{c}{2}, 0)$ is a point on this orbit and thus on equation [22] (with $C = D = 0$). If we take the initial conditions at $s = 0$ to be $\phi(0) = \frac{c}{2}, \phi'(0) = 0$, then it follows that $K = 0$, which gives us equation [26]. Since $\phi \geq 0$ by hypothesis, equation [26] reduces to equation [27]. The solution is then expressed by equation [28]. This gives us the **soliton solution** that is shown in equation [29].

3 Classification of Waves

There does not seem to be a precise definition of what exactly constitutes a wave. One can give various restrictive definitions, but to cover the whole range of wave phenomena it seems preferable to be guided by the intuitive view that a wave is any recognizeable signal that is tranferred from one part of a medium to another with some recognizeable velocity of propagation.

The signal may take many forms, but one can still two main classes of waves. The first is formulated mathematically in terms of hyperbolic differential equations, and such waves will be termed **hyperbolic**. The second class is less easily characterized. However, it includes the simplest cases of dispersive waves in linear problems, so such waves are known as **dispersive**. Note that there are some waves which are classified as both of the above types of waves.

3.1 Hyperbolic Waves

The prototype for hyperbolic waves is taken to be the wave equation $\phi_t t = c^2 \Delta\phi$ (equation [30]), where Δ represents the Laplacian operator. Hyperbolic equations can be defined in a manner that depends only on the form of the equation and is independent of whether explicit solutions can be obtained or not. However, the same cannot be said of dispersive waves.

The wave equation is linear, so it is clearly does not provide a complete picture of hyperbolic waves. The most outstanding new phenomenon of the theory of nonlinear hyperbolic equations is the appearance of shock waves, which can be seen physically as discontinuities in physical quantities like pressure, density, and velocity. For example, the blast waves of explosions and the sonic booms of high speed aircraft are examples of shocks. Indeed, the entire machinery of nonlinear hyperbolic differential equations had to be devoloped to explain and predict such phenomena. The theory builds up linear equations to **quasi-linear** and ultimately to fully nonlinear coupled hyperbolic equations. See [3] for an extensive study of hyperbolic waves.

3.2 Dispersive Waves

Dispersive waves are considerably more difficult to classify than hyperbolic ones. The protoype for a dispersive wave is based on the form of the solution rather than that of the equation. A linear dispersive system, for example, is any system that admit solutions of the form $\phi(x,t) = a\cos(\kappa x - \omega t)$ (equation [31]), where the frequency ω is a real-valued function of the wave number κ. The phase speed is then $\frac{\omega(\kappa)}{\kappa}$ and waves are said to be **dispersive** if this quantity is not a constant. The terminology refers to the fact that the solution will consist of the superposition of several modes with different values for κ. If the phase speed is not the same for all phases, the modes with different κ will propagate at different speeds. This will cause the wave to "disperse." The definition can be modified somewhat to say that a wave is dispersive if $\omega'(\kappa)$ is not a constant.

The discussion of the classification of dispersive waves stems from certain types of oscillatory solutions representing a wave train. Such solutions canbe obtained from a wide variety of partial differential equations and several integral equations as well. It is the disperson relation $\omega = W(\kappa)$ that characterizes the problem. The source of this relation is quite important. The KdV equation ($\phi_t + c_0\phi_x + \nu\phi_{xxx} = 0$, equation [32]), for example, is a dispersive wave with dispersion related given by $\omega = c_0\kappa - \nu\kappa^3$ (equation [33]).

We obtain solutions for dispersive waves that are more general than the form in equation [31] by superposition to form Fourier integrals. (We cannot just restrict ourselves to summation because some dispersive equations lead to singular eigenvalue problems.) Formally, one can then find using the Fourier integral theorem a solution that fits the given initial and boundary conditions. This solution takes the form shown in equation [34], where $W(x)$ is the dispersion relation, and we can (formally) find a solution for arbitrary $F(x)$. We can study the dispersion of these waves by various asymptotic expansions of equation [34].

A key concept that arises from such analysis is that of **group velocity** $C(\kappa) = \frac{dW}{d\kappa}$. The oscillatory wave train arising from equation [34] has a variable wavelength. The different values of the wave number propagate through this oscillatory train, and the speed of propagation is the group velocity. One also finds that the energy propagates with the group velocity.

Let us now consider the KdV equation as an example of nonlinear dispersion. In 1895, Korteweg and de Vries showed that long waves in (relatively) shallow water could be described accurately by the nonlinear equation $\eta_t + (c_0 + c_1\eta)\eta_x + \nu\eta_{xxx} = 0$ (equation [35]), where c_0, c_1, and η are constants. If we linearize the equation by considering the small amplitude case, the case $c_1\eta\eta_x$ term drops out. The resulting linear equation has the solutions $\eta = a\cos(\kappa x - \omega t), \omega = c_0\kappa - \nu\kappa^3$ (equation [36]). If one expands the amplitude as Stokes did in 1847 (see [3] for this analysis), one can give an improved approximation to the solution of the original nonlinear equation. Korteweg and de Vries showed that periodic solutions $\eta = f(\theta), \theta = \kappa x - \omega t$ of equation [35] could be found in closed form in terms of Jacobian elliptic functions. Because $f(\theta)$ was found in terms of the elliptic function $\mathrm{cn}(\theta)$, they termed the solutions **cnoidal** waves. The results of their work is consistent with Stokes' conclusion that the surface elevation η in a planar wavetrain on deep water can be expanded in powers of the amplitude a. In the case of the KdV equation, the existence of periodic wavtrains is demonstrated explicitly. Additionally, $f(\theta)$ contains an arbitrary amplitude a, and the solution includes a specified dispersion relation between ω, κ, and a. Note that the most important nonlinear effect is the inclusion of the amplitude in the dispersion relation.

The analysis did not end there. The limit of $\mathrm{cn}(\theta)$ as the modulus tends to 1 is the hyperbolic secant. This gives us the solution described by equation [37]. In the limit, the period of the solution has become infinite, and the solution η represents a single hump of positive elevation. It is the "solitary wave" that was discovered experimentally by Scott Russell in 1844. It was previously analyzed on an approximate basis by Boussinesq in 1871 and by Rayleigh in 1876. The inclusion of the solitary wave with the periodic wavetrains in the same analysis was a significant step. The equation for the velocity of propagation U in terms of the amplitude is the remnant of the dispersion relation in this nonperiodic case.

The KdV equation originated in water waves, but it was later realized that it is one of the simplest prototypes that combines nonlinearity and dispersion. In this respect, it is analogous to Burgers' equation, which combines nonlinearity with diffusion. The KdV equation has now been derived as a useful equation in other fields.

In recent years, other simple equations have been derived and also used as prototypes to develop and test ideas. Two examples of these are a generalization of the linear Klein-Gordon equation (equation [38]) and a generalization of Schrodinger's equation (equation [39]). The Sine-Gordon equation—an example of equation [38] in which $V'(\phi) = \sin(\phi)$—and equation [39] both share with the

KdV equation the property of having solitary wave solutions as limiting cases. Solitary waves are obviously of interest, as they are strictly nonlinear phenomena with no counterparts in linear dispersive theory. Their behavior in a highly nontrivial manner. For example, solitary waves retain their individuality under interaction and eventually emerge with their original shapes and speeds. Moreover, these solutions are but one class obtained in a general analytical approach on these equations. There are further results concerning the solutions of these equations under arbitrary initial conditions. See [3] for much of this analysis.

4 Water Waves

4.1 Introduction

Many of the general ideas about dispersive waves originated in the problems of water waves. Consider an inviscid incompressible fluid (such as water) in a constant gravitational field. Denote the spacial coordinates by (x_1, x_2, y) and the corresponding velocity vector u by (u_1, u_2, v). The gravitational acceleration g is in the negative y direction. Also assume that the density ρ is constant and that there is an external force $F = -\rho g\hat{\jmath}$, where $\hat{\jmath}$ denotes the unit vector in the y direction. The equations are written as formulas [40] and [41]. In the case of water waves, one can assume that the fluid is irrotational (in other words, that $\text{curl}(u) = 0$) and that $u = \nabla\phi$ for some velocity potential ϕ. This gives us the Helmholtz equation (equation [42]), where $\omega = \text{curl}(u)$ is the vorticity. One can use a variational formulation to find equation [43] for the velocity potential ϕ. In this equation, $y = -h_0(x_1, x_2)$ is the ocean floor, and $y = \eta(x_1, x_2, t)$ is the surface.

4.2 Linearized Water Waves

The above analysis can be continued quite extensively (see chapter 13 of [3]), but let us consider the case of linearized water waves. For small perturbations on water initially at rest, ϕ and η are small and we can justifiably linearize the equations to obtain a leading order analysis. In the case of water waves, the waves propagate horizontally in the sense that the elementary sinusoidal solutions take the form in equation [44]. Note that these two formulas are oscillatory in x and t but not in y. Extensive analysis in this case is also possible, but I will let the interested reader see the details in [3]. One important point is the dispersion related (equation [45]) that one obtains as a result of this analysis.

The dispersion relation has two modes $\omega = \pm W(\kappa)$, represented by the positive and negative square roots. Note that there is not a branch point at $\kappa = 0$, though on a first glance it may seem as though there is one. W does have branch points, however, and the other zeros and infinities of the $\tanh(\kappa h_0)$. These are $\kappa h_0 = \pm n\pi i$ and $\kappa h_0 = \pm(n - \frac{1}{2}\pi i$, where n is a natural number. The functions $W(\kappa)$ and $-W(\kappa)$ are both single valued analytic functions of κ in the complex plane cut from $-\infty i$ to $-\frac{\pi i}{2h_0}$ and from $\frac{\pi i}{2h_0}$ to ∞i.

The general solution is obtained by Fourier transforming equation [46] with two modes corresponding to $\omega = \pm W(\kappa)$. Two initial conditions are needed to determine the arbitrary functions $F(\kappa)$ that appear in the transforms. Additionally, any prescribed function ϕ must satisfy Laplace's equation, because otherwise compressibility effects will come into play and change the initial distribution rapidly to some new effective initial distribution. For a delta function initial condition, the solution is given by equation [47]. One can then analyze the asymptotic behavior of the one dimensional solution using the method of stationary phase.

4.3 Shallow Water Theory

For "gravity waves" with $\kappa h_0 \longrightarrow 0$, the dispersion relation is approximately $\omega^2 \sim g h_0 \kappa^2$ (equation [48]), and the phase speed c_0 given by equation [49] becomes independent of κ. The Fourier superposition effects drop out, and we obtain the d'Alembert solution of the wave equation $\eta = f(x - c_0 t) + g(x + c_0 t)$ (equation [50]). Dispersion effects do not appear in this approximation. We will soon analyze shallow water waves as the first terms in the expansion of $(\frac{h_0}{l})^2$. By going to the next order, we will pick up small dispersive effects. The linear equations lead to the wave equation, but one can obtain a hyperbolic system of partial differential equations using the methods in Part I of [3]. In the one-dimensional case, we may also decouple the equations using the method of Riemann Invariants. We obtain what is called a turbulent **bore**. This breaking phenomenon is one of the most interesting problems in water wave theory. When gradients are no longer small, the approximation $\frac{h_0^2}{l^2}$ is no longer valid, so the solution one obtains under this approximation cease to apply long before breaking occurs. In fact, shallow water wave theory goes too far; it predicts that all waves that carry an increase of elevation eventually break, but empirical evidence has established that some waves never break. It is the dispersion effects that we have so far neglected that inhibit breaking.

Let us now incorporate these dispersion effects into our analysis. One can do this by more formally expanding in the small parameter $(\frac{h_0}{l})^2$ and including one more term in this expansion than we did previously. Consider the one-dimensional case where h_0 is a constant. If one restricts oneself to waves moving only to the right, one obtains the KdV equation. For waves moving only to the right, the first two terms in the dispersion relation are $\omega = c_0\kappa - \gamma\kappa^3$ and $\gamma = \frac{1}{6}c_0 h_0^2$, which correspond to the equation $\eta_t + c_0\eta_x + \gamma\eta_{xxx} = 0$. In the nonlinear shallow water wave equations (equation [51]), waves moving to the right in undisturbed water of depth h_0 satsify the Riemann invariant given by equation [52]. After a few lines of grunge, we find that if we approximate the nonlinear terms to first order in $\frac{a}{h_0}$, one obtains the Korteweg-de Vries equation (equation [53]). The linearized equation has a disperion relation given by equation [54], which is consistent with what we obtained previously when κ is small and has bounded phase and group velocities when κ is large.

The KdV equation does not apply only to water waves. It has, for example, arisen in plasma physics. Moreover, it can be derived in other manners. For example, it can be derived via a perturbation expansion. One can consider any system in which the Boussinesq equation describes a wave and obtain the KdV equation by specifying that waves must be travelling to the right. See [3] for examples of such analysis. Whitham's text also includes a solution of the KdV equation. It takes the form given in equation [55], where the modulus s of the standard Jacobian elliptic function is given by equation [56], and the wavelength λ is given by equation [57]. In the above equations, $K(s)$ is the complete elliptic integral of the first kind and $X = x - Ut$. Additionally, $\alpha = \frac{a}{h_0}$ and $\beta = \frac{h_0^2}{l^2}$. The modulus s measures the relative importance of nonlinearity to dispersion. In the linear limit as $s \longrightarrow 0$, $\mathrm{cn}(\xi) \longrightarrow \cos(\xi)$, whereas in the solitary limit as $s \longrightarrow 1$, $\mathrm{cn}(\xi) \longrightarrow \mathrm{sech}(\xi)$. Note that the cnoidal waves are solutions of the KdV equation subject only to the restriction that $0 \leq \alpha \leq \beta$. However, the equations are valid only when the parameters α and β are small. Like solitary waves, they obtain a maximum height when crests peak.

5 Conclusion

The analysis in this paper is far from exhaustive, as there is much more to be said about the Korteweg-de Vries equation. One can, for example, compare and contrast KdV, Boussinesq, and Stokes' waves. They are found with similar analysis, but some of their properties are strikingly different. Additionally, one can analysis the breaking and peaking of waves. The nonlinear shallow water equations that neglect dispersion lead to the breaking associated with hyperbolic systems. In such breaking, the solution develops a vertical slope and a mulitvalued profile. Because of dispersion, this does not occur in solutions to the KdV equation. One instead obtains solitary and periodic waves that are not found in nondispersive shallow water theory.

One can also add a second derivative term to the KdV equation and analyze the structure of bores. One can justify the addition of this term because it can represent dissipation. This term is necessary because the KdV equation has no solutions that propogate without changing shape. Yet another important discussion involves interacting solitary waves. The numerical analysis of the KdV equation is also extremely important. With the exception of the numerics, these topics are covered extensively in [3]. The interested reader is invited to pursue this subject further.

Bibliography

[1] Kath, Bill, and Kath, Bonnie, 1998. Making Waves: Solitons and Their Optical Applications, *Siam News*, **Volume 31, Number 2**, 1, 12-13.

[2] Marsden, Jerrold E., and Ratiu, Tudor S., 1998 (so far unpublished). *Introduction to Mechanics and Symmetry*, Second Edition. New York, NY: Springer Verlag.

[3] Whitham, Gerald B., 1974. *Linear and Nonlinear Waves.* New York, NY: John Wiley and Sons.

Mason Porter
CDS 241a
3/14/98

Formulas for Final Project

[1] $\Omega(u_1,u_2) = \frac{1}{2}\left(\int_{-\infty}^{\infty}[\hat{u}_1(x)u_2(x) - \hat{u}_2(x)u_1(x)]\,dx\right)$ (equ 3.2.9 in [1])

← all these should be [2]

[2] $\int_{-\infty}^{\infty}\hat{u}_2(x)u_1(x)\,dx = -\int_{-\infty}^{\infty}\hat{u}_1(x)u_2(x)\,dx$

[3] $\Omega(u_1,u_2) = \int_{-\infty}^{\infty}\hat{u}_1(x)u_2(x)\,dx = \int_{-\infty}^{\infty}v(x)u_2(x)\,dx$ (3.2.10 in [1])

[4] $\Omega(w,v) = \Omega\left(\frac{\partial v}{\partial x}, v\right) = \int_{-\infty}^{\infty}(v(x))^2\,dx \neq 0$

[5] $X_H(u) = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right)$ (3.2.11 in [1])

[6] $\Omega(X_H(v),w) = \int_{-\infty}^{\infty}\frac{\delta H}{\delta v}(x)w(x)\,dx = dH(v)\cdot w$

[7] $u_t = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right)$ (3.2.12 in [1])

[8] $H_1(u) = -\frac{1}{6}\int_{-\infty}^{\infty}u^3\,dx$

[9] $u_t + uu_x = 0$ (3.2.13 in [1])

[10] $H_2(u) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_x^2 - u^3\right)dx$ (3.2.14 in [1])

[11] $u_t + 6uu_x + u_{xxx} = 0$ (3.2.15 in [1])

2)

MAP
CDS 241a
3/14/98

Formulas

[12] $\{F, G\}(z) = \Omega(X_F(z), X_G(z))$ (2.7.1 in [1])

[13] $\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u} \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right) dx$ (3.3.4 in [1])

[14] $F_0(u) = \int_{-\infty}^{\infty} u \, dx$

[15] $F_1(u) = \int_{-\infty}^{\infty} \frac{1}{2} u^2 \, dx$

[16] $F_2(u) = \int_{-\infty}^{\infty} \left(-u^3 + \frac{1}{2}(u_x)^2\right) dx$

[17] $F_3(u) = \int_{-\infty}^{\infty} \left(\frac{5}{2}u^4 - 5uu_x^2 + \frac{1}{2}(u_{xx})^2\right) dx$

[18] $c\phi' - 6\phi\phi' - \phi''' = 0$ (3.2.16 in [1])

[19] $c - 3\phi^2 - \phi'' = C$ (3.2.17 in [1])

[20] $h(\phi, \phi') = \frac{1}{2}(\phi')^2 - \frac{c}{2}\phi^2 + \phi^3 + C\phi$ (3.2.18 in [1])

[21] $\phi' = \pm\left(c\phi^2 - 2\phi^3 - 2C\phi + 2D\right)^{1/2}$ (3.2.19 in [1])

[22] $s = \pm \int \frac{d\phi}{\sqrt{c\phi^2 - 2\phi^3 - 2C\phi + 2D}}$ (3.2.20 in [1])

MAP
CDS 241a
3/14/98

Formulas

[23] $s = \pm \int \frac{d\phi}{\sqrt{c\phi^2 - 2\phi^3}} = \pm \frac{1}{\sqrt{c}} \log \left| \frac{\sqrt{c-2\phi} - \sqrt{c}}{\sqrt{c-2\phi} + \sqrt{c}} \right| + K$ (3.2.21 in [1])

[24] $h(\phi, \phi') = \frac{1}{2}(\phi')^2 - \frac{c}{2}\phi^2 + \phi^3$ (3.2.22 in [1])

[25] $\begin{bmatrix} 0 & 1 \\ \pm c & 0 \end{bmatrix}$

[26] $\left| \frac{\sqrt{c-2\phi} - \sqrt{c}}{\sqrt{c-2\phi} + \sqrt{c}} \right| = e^{\pm\sqrt{c}\, s}$

[27] $\frac{\sqrt{c-2\phi} - \sqrt{c}}{\sqrt{c-2\phi} + \sqrt{c}} = -e^{\pm\sqrt{c}\, s}$

[28] $\phi(s) = \frac{2c e^{\pm\sqrt{c}\, s}}{(1+e^{\pm\sqrt{c}\, s})^2} = \frac{c}{2\cosh^2(\sqrt{c}\, s/2)}$

[29] $u(x,t) = \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct)\right]$

[30] $\phi_{tt} = c_0^2 \Delta\phi$ (1.1 in [2]) ← all these should be [3]

[31] $\phi(x,t) = a\cos(kx - \omega t)$ (1.3 in [2])

[32] $\phi_t + c_0\phi_x + \nu\phi_{xxx} = 0$ (part of 1.19 in [2])

[33] $\omega = c_0 k - \nu k^3$ (part of 1.19 in [2])

4

MAP
CDS 241a
3/15/98

Formulas

[34] $\phi = \int_0^\infty F(k)\cos(kx - Wt)\,dk$ (1.23 in [2])

[35] $\eta_t + (c_0 + c_1\eta)\eta_x + \nu\eta_{xxx} = 0$ (1.36 in [2])

[36] $\eta = a\cos(kx - \omega t),\ \omega = c_0 k - \nu k^3$ (1.37 in [2])

[37] $\eta = a\,\mathrm{sech}^2\left[\left(\frac{c_1 a}{12\nu}\right)^{1/2}(x - Ut)\right],\ U = c_0 + \frac{1}{3}c_1 a$ (1.38, 1.39 in [2])

[38] $\phi_{tt} - \phi_{xx} + V'(\phi) = 0$ (1.40 in [2])

[39] $i\psi_t + \psi_{xx} + |\psi|^2\psi = 0$ (1.41 in [2])

[40] $\nabla\cdot u = 0$ (13.1 in [2])

[41] $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\frac{1}{\rho}\nabla p - g\hat{j}$ (13.2 in [2])

[42] $\frac{\partial\omega}{\partial t} + \nabla\times(\omega\times u) = 0$ (13.4 in [2])

[43] (13.19 in [2])

$\phi_{x_i x_i} + \phi_{yy} = 0 \quad (-h_0 < y < \eta)$

$\eta_t + \phi_{x_i}\eta_{x_i} - \phi_y = 0 \quad (y = \eta)$

$\phi_{x_i} h_{0x_i} + \phi_y = 0 \quad (y = -h_0)$

MAP
CDS 241a
3/16/98

Formulas

[44] $\eta = Ae^{ik\cdot x - i\omega t}$, $\phi = Y(y)e^{ik\cdot x - i\omega t}$

[45] $\omega^2 = gk \tanh(kh_0)$ (13.25 in [2])

[46] $\eta = Ae^{ik\cdot x - \omega t}$

$\phi = \frac{-ig}{\omega} A \frac{\cosh k(h_0 + y)}{\cosh kh_0} e^{ik\cdot x - \omega t}$ (13.24 in [2])

[47] $\eta(r,t) = \frac{1}{2\pi}\int_0^\infty k J_0(kr)\cos(w(k)t)\,dk$ (13.33 in [2])

[48] $\omega^2 \sim gh_0 k^2$ (13.73 in [2])

[49] $c_0 = \sqrt{gh_0}$

[50] $\eta = f(x - c_0 t) + g(x + c_0 t)$

[51] $h_t + (uh)_x = 0$ (13.79 in [2])

$u_t + uu_x + gh_x = 0$

[52] $u = 2\sqrt{g(h_0 + \eta)} - 2\sqrt{gh_0}$ (13.96 in [2])

(6)

MAP
CDS 241a
3/16/98

Formulas

[53] $\eta_t + c_0\left(1 + \frac{3}{2}\frac{\eta}{h_0}\right)\eta_x - \gamma\eta_{xxx} = 0$ (13.99 in [2])

[54] $\omega = \dfrac{c_0 k}{1 + \gamma k^2/c_0}$

[55] $s = \alpha\, cn^2 \left(\frac{3\beta}{4h_0^2}\right)^{1/2} X$ (13.114 in [2])

[56] $s = \left(\frac{\alpha}{\beta}\right)^{1/2}$ (13.115 in [2])

[57] $\lambda = \frac{4h_0}{\sqrt{3\beta}} K(s)$ (13.116 in [2])