Stability of Hamiltonian Systems: Energy-Momentum Method and Effect of Dissipation

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Excellent report. Your remarks on the heavy top are very insightful! A+ JM

In this report, I review the results presented by Bloch, Krishnaprasad, Marsden and Ratiu (1994a, 1994b) on the effect of dissipation on formally unstable Hamiltonian systems. As we shall see, for the Hamiltonian systems which are formally unstable, *i.e.* the second variation of the Hamiltoninan, or the reduced Hamiltonian, is indefinite, introduction of small dissipation results in nonlinear and spectral instability. The added dissipation may affect the internal variables (the variables which are invariant under the group action), in which case the dissipation term can be regarded as the gradient of a Rayleigh dissipation function with respect to the reduced phase-space. Alternatively, the dissipation term may directly be added to the group or Lie algebra variables. We see that in order to maintain conservation of momentum, this has to be of a "double bracket" form similar to what considered by Brockett (1993). This term can again be regarded as the gradient of a Rayleigh dissipation function but tangent to the coadjoint orbits, where in case of rigid body for example the magnitude of the angular momentum is preserved. We apply and compare these results for the heavy top example.

1. Introduction.

In this review, we consider the issue of stability of Hamiltonian systems. For the sake of simplicity, we restrict our discussion to finite-dimensional systems, although under certain conditions of regularity and for the cases where the linearized system has a discrete spectrum, these results extend to the infinite-dimensional case as well. In fact, these results are probably more interesting and applicable for infinite-dimensional systems.

To study stability of equilibria of nonlinear systems, one either uses spectral information about the linearized Hamiltonian system or attempts to find a Lyapunov function in order to establish Lyapunov stability of the system.

Now, in the case of Hamiltonian systems on symplectic manifolds, it is well-known that the spectrum (set of eigenvalues) of the linearized system, in addition of being symmetric with respect to the real axis, is also symmetric with respect to the imaginary axis. Let

$\dot{z} = X_H(z)$ with respect to the Hamiltonian $H : P \to \mathcal{R}$ represent the dynamics of the Hamiltonian system. Then, the above fact is a direct result of $DX_H(z)$ being Ω-skew or infinitesimally symplectic, which in local coordinates is equivalent to saying

$$\Omega\, DX_H(z) + (DX_H(z))^T\, \Omega = 0.$$

See for example Abraham and Marsden (1978). Thus, unless a Hamiltonian system is spectrally and nonlinearly unstable because the spectrum has components in the open right and left half-planes of the complex plane, the spectrum must lie on the imaginary axis, in which case spectral analysis alone can not determine stability of the system.

Alternatively, one can use Lyapunov method to investigate stability. It is well-known that if there exists a Lyapunov function $W : P \to \mathcal{R}$, which is positive definite with respect to the equilibrium point z_e $(\delta\, W(z_e) = 0,\ \ \delta^2 W(z_e) > 0)$, and has negative time-derivative along the trajectories of the system in a neighborhood of z_e $(\dot{W}(z) \leq 0)$, then z_e is nonlinearly stable. Conversely, existence of such a function W, such that $\dot{W}(z) < 0$ in a neighborhood of z_e, but $\delta^2 W(z_e)$ is indefinite, is sufficient to show that the equilibrium is nonlinearly unstable. Since the Hamiltonian is a conserved quantity for the Hamiltonian system $(\dot{H}(z) = 0)$, one candidate for the Lyapunov function is H (or $-H$). $dH(z) \cdot v = \Omega(X_H, v)$ on a symplectic manifold, implying that any equilibrium of the Hamiltonian dynamics is a **critical point** of $\delta\, H$, *i.e.* $\delta\, H(z_e) = 0$. Therefore, if $\delta^2 H$ is positive definite (or negative definite) at some equilibrium, that point is stable in the sense of Lyapunov.

However, if $\delta^2 H(z_e)$ is indefinite or semidefinite, one can not say anything about stability of z_e. Notice that to prove instability one needs $\dot{H} < 0$ or $\dot{H} > 0$.
In many cases though, $\delta^2 H(z_e)$ is indefinite or semidefinite and thus, one must use other techniques or Lyapunov function candidates. For example, if C is another conserved quantity $(\dot{C}(z) = 0)$, one can consider C or $H + C$, such that z_e is a critical point of $\delta\, C$ or $\delta\,(H + C)$. This is the basis of energy-Casimir method, where one uses appropriate Casimir functions, which are conserved quantities depending only on the Poisson structure of the system and independent of the particular Hamiltonian. Unfortunately, for symplectic structures the only Casimirs are constant, and in general there exists no systematic method to find these functions.

To further complicate the issue, we are often interested in examining the stability of certain orbits or trajectories generated by the action of a symmetry group G, which we call relative equilibria. Now, if the Hamiltonian dynamics are invariant with respect to the action of G, they induces a reduced system on P/G, where $[z_e]$ the reduction of z_e to the quotient is an equilibrium point. Therefore, one can investigate the stability of the reduced Hamiltonian system on P/G instead. It is obvious that for a compact symmetry group, if $[z_e]$ is a stable equilibrium for the reduced system, then z_e is a stable relative equilibrium for the original system. But P/G is not usually symplectic and in the non-symplectic case, the equilibrium may not be a critical point of $\delta\, H$.

Alternatively, one can examine the reduced Hamiltonian system and variations of H on the symplectic leaves which is the basis for the energy-momentum methods.

To summarize the above points, we have:

- **Nonlinear stability (Stability in the sense of Lyapunov):** We say a trajectory $\{z_t\ t \geq 0\}$ is stable in the sense of Lyapunov if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\overline{z}_0 - z_0\| \leq \delta$, $\sup_t \|z_t' - z_t\| \leq \epsilon$ where $\overline{z}_t$ is the trajectory starting at $\overline{z}_0$. If the statement holds only in a neighborhood of the trajectory, then it is **locally stable**.
- **Spectral stability (instability):** An equilibrium is spectrally stable (unstable) if the spectrum of the linearized system with respect to that equilibrium lies in the closed left half-plane (has components in the open right half-plane) of the complex plane. Spectral instability implies nonlinear instability, but for the Hamiltonian systems, the converse is not true.
- **Formal Stability (instability):** An equilibrium of a Hamiltonian system is formally stable (unstable) if it is a critical point of the first variation of some conserved quantity (δH for example), while the second variation is definite (indefinite). Formal stability implies nonlinear stability, but not vice versa.
- **Lyapunov stability (instability):** An equilibrium is locally Lyapunov and thus, nonlinearly stable if there exists a Lyapunov function $W : P \rightarrow \mathcal{R}$ such that $\delta W(z_e) = 0$, $\delta^2 W(z_e) > 0$ or < 0, and $\dot{W}(z_t) \leq 0$ in a neighborhood of z_e.
  The equilibrium is Lyapunov and nonlinearly unstable if W is as above, but $\delta^2 W(z_e)$ is indefinite and $\dot{W}(z_t) < 0$ in a neighborhood of z_e.

If we perturb a Hamiltonian system by introducing small dissipation into the system such that the equilibria are preserved, the above implications may no longer be true. This perturbation may cause the eigenvalues of a spectrally unstable system to move into the left half-plane, or drive some of the eigenvalues of a spectrally stable system into the right half-plane. In the former case the system is nonlinearly stable, while in the latter it is nonlinearly unstable. In fact, we show that if $\delta^2 H$ is indefinite, addition of dissipation results in the latter case.

Addition of small dissipation to the Hamiltonian system results in $\dot{H} \leq 0$ with $\dot{H}$ being strictly negative in certain directions. If $\delta^2 H(z_e)$ is positive definite, then the equilibrium is still stable after introducing the dissipation. On the other hand, if $\delta^2 H$ is indefinite, we will see that in certain situations, there exists a perturabation of H, W_H such that $\dot{W}_H < 0$ in some neighborhood of z_e, while $\delta W_H(z_e) = 0$ and $\delta^2 W_H(z_e)$ is indefinite, meaning that the perturbed system is nonlinearly unstable.

In fact, what we show is that the time-derivative of $W_H(\delta z_t)$ "with respect to the linearized system" is negative. Here, δz_t denotes the integral curves of the approximate linearized system;

$$\dot{\delta z} = DX_H(z_e)\delta z, \qquad \text{and} \qquad \dot{W}_H = dW_H \cdot DX_H(z_e)\delta z.$$

This of course implies $\dot{W}_H < 0$ in a neighborhood of z_e. But it also results in the stronger implication that the linearized system is spectrally unstable.
To see this, note that the spectrum can not entirely be in the open left half-plane, otherwise the equilibrium is stable. It can not also have zero components, in which case $\dot{W}_H(\delta z_t) = 0$ in some subspace invariant with respect to trajectories of δz_t. One can also show that no eigenvalue lies on the imaginary axis, otherwise there must exist an invariant subspace, in which the trajectories of δz_t are periodic. But then, $W_H(\delta z_t)$ along those trajectoris must be periodic which contradicts $\dot{W}_H(\delta z_t) < 0$. Therefore, some eigenvalues must lie in the open right half-plane.

Example 1. To clarify our discussion, let us consider the following simple linear Hamiltonian system on the symplectic space $(\mathcal{R}^{2n}, \Omega_S)$, where M is a positive definite $n \times n$ matrix, B is a skew-symmetric matrix and Σ is a symetric matrix. One can regard this system as a reduced linearized Hamiltonian system with respect to an Abelian symmetry group, where all variables can be considered as **internal variables**:

$$M\,\ddot{q} \;+\; B\,\dot{q} + \Sigma\,q = 0, \quad q \in \mathcal{R}^n \qquad \Longleftrightarrow \qquad \begin{cases} \dot{p} = -B\,M^{-1}p - \Sigma\,q \\ \dot{q} = M^{-1}p. \end{cases} \tag{1}$$

The system is clearly Hamiltonian with respect to the following structure:

$$H(q,p) = \frac{1}{2}p \cdot M^{-1}p \;+\; \frac{1}{2}q \cdot \Sigma q, \qquad \Omega_S = \begin{bmatrix} -B & I \\ -I & 0 \end{bmatrix}. \tag{2}$$

The only equilibrium for this system is $z_e = 0$. If Σ is positive definite, then $\delta^2 H(0) = \mathrm{Diag}(M^{-1}, \Sigma)$ is positive definite and thus, z_e is stable. In this case, all eigenvalues lie on the imaginary axis.
If Σ has some negative eigenvalues then $\delta^2 H(0)$ is indefinite and the system may or may not be stable. For certain choices of Σ, one can in fact show that the spectrum lies on the imaginary axis and 0 is spectrally stable.

Adding a certain small dissipation to the above system results in the following dynamics, where R is assumed to be positive definite and $\epsilon > 0$ is small:

$$M\,\ddot{q} \;+\; (B + \epsilon\,R)\,\dot{q} + \Sigma\,q = 0, \qquad \Longleftrightarrow \qquad \begin{cases} \dot{p} = -(B + \epsilon\,R)\,M^{-1}p - \Sigma\,q \\ \dot{q} = M^{-1}p. \end{cases} \tag{3}$$

For the perturbed system, one finds that

$$\dot{H} = -\epsilon\,M^{-1}p \cdot R\,M^{-1}p \leq 0.$$

The above equations can be rewritten as

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -B\,M^{-1}p - \Sigma q \\ M^{-1}p \end{bmatrix} - \epsilon \begin{bmatrix} R\,\dot{q} \\ 0 \end{bmatrix} = X_{_H} - \epsilon \begin{bmatrix} \frac{\delta}{\delta \dot{q}} \left(\frac{1}{2}\dot{q} \cdot R\,\dot{q} \right) \\ 0 \end{bmatrix}, \tag{3'}$$

where the function $\mathcal{R}(\dot{q}) = \frac{1}{2}\dot{q} \cdot R\,\dot{q}$ is called a **Rayleigh dissipation function** and $F = -R\,\dot{q}$ a **dissipative force field.**
If Σ is positive definite, one can still use H as Lyapunov function to show that 0 is stable. Also, the corresponding eigenvalues will be in the closed left half-plane.

However, H is not an adequate choice if Σ has negative eigenvalues. Nevertheless, one can find in this case a perturbation of H

$$W_{_H} = H \;+\; M^T K^{-1} \Sigma\, q \cdot M^{-1} p,$$

such that for some sufficiently small β and positive definite K, $\dot{W}_{_H} < 0$. Since $\delta^2 W_{_H}(0)$ is indefinite for sufficiently small β, the above means that 0 is unstable. As mentioned above it also implies spectral instability. Thus, we have:

$\triangle$ *For the perturbed Hamiltonian system (3), the equilibrium remains nonlinearly and spectrally stable with addition of small Rayleigh dissipation term, if $\delta^2 H(0)$ or Σ is positive definite. On the other hand, addition of the dissipation term results in a nonlinearly and spectrally unstable equilibrium, if $\delta^2 H(0)$ is indefinite or Σ has negative eigenvalues.* $\square$

We generalize the above result in the next sections to the cases where group variables exist, and where the dissipation term affects the group variables instead and then continue with some examples. But first, we briefly review the energy-momentum method.

2. The Energy-Momentum Method

Let (T^*Q, Ω) be a symplectic manifold, with the Lie group G acting properly and freely on T^*Q through the (left) action Φ_g for all $g \in G$, and the standard momentum map $\mathbf{J} : T^*Q \to \mathcal{G}^*$. Consider the Hamiltonian dynamics corresponding to the Φ_g-invariant Hamiltonian H. Then there is a natural reduction of the Hamiltonian system to T^*Q/G, where the symplectic structure reduces to a Poisson structure on T^*Q/G.

Furthermore, using the Noether's theorem; *i.e.* conservation of the momentum map along the trajectories of the Hamiltonian system, one can further reduce the dynamics to the symplectic leaf $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, where $\mu \in \mathcal{G}^*$ is the initial momentum of the system and a regular value of $\mathbf{J}$, and G_μ is the symmetry or invariance group of $(H, \mathbf{J})$. Using the mechanical connection ϱ, the reduced Hamiltonian structure on P_μ is given by the Hamiltonian $h_\mu = H_\mu(q, p{+}\alpha_\mu)$, $\alpha_\mu = \langle \varrho, \mu \rangle$, dropped to the quotient, and the symplectic form Ω_μ on $T^*(Q/G_\mu)$ restricted to the tangent space of P_μ.

We would like to study the stability of the relative equilibria of the above Hamiltonian system. Let $\{z(t, z_e), t \geq 0\}$ denote the integral curve of X_H passing through z_e at $t = 0$. Then, z_e or the entire trajectory $z(t, z_e)$, is called a **relative equilibrium** if for all $t \geq 0$,

$$z(t, z_e) = \Phi_{\exp(t\xi_e)} z_e, \qquad \text{for some} \qquad \xi_e \in \mathcal{G}_{\mu_e} \qquad \text{where} \qquad \mu_e = \mathbf{J}(z_e) = \mathbf{J}(z(t, z_e)). \tag{4}$$

It is easy to see that z_e is a critical point of the G_μ-invariant augmented Hamiltonian H_μ. One can further show that at the relative equilibrium, $\mu_e = \mathbb{I}(q_e)\xi_e$ and

$$\alpha_{\mu_e}(q_e) = \mathbb{F}L\left(\left(\xi_e\right)_Q(q)\right) \qquad \text{where} \qquad z_e = (q_e, p_e). \tag{4'}$$

$[z_e] \in P_{\mu_e}$, the reduction of $z(t, z_e)$ to the quotient with respect to G_{μ_e}, is obviously an equilibrium of the reduced system on P_{μ_e} and a critical point of δh_{μ_e}. Clearly, if $[z_e]$ is a stable equilibrium point for the reduced Hamiltonian structure, the relative equilibrium is stable with respect to the original Hamiltonian. Thus, it suffices to study the stability of $[z_e]$ in the reduced Hamiltonian dynamics.

In the energy-momentum method, one considers $\delta^2 H_{\mu_e}(z_e)$ along variations tangent to $\mathbf{J}^{-1}(\mu_e)$ and transverse to $T_{z_e}(G_{\mu_e} \cdot z_e)$, where

$$G_{\mu_e} \cdot z_e = \{\Phi_{\exp(t\xi)} z_e, \ \forall \xi \in \mathcal{G}_{\mu_e}\}$$

is the orbit of G_{μ_e} through z_e. Since $\delta^2 H_{\mu_e}$ along $T_{z_e}(G_{\mu_e} \cdot z_e)$ is zero, it turns out that $\delta^2 h_{\mu_e}([z_e])$ has the same spectrum as $\delta^2 H_{\mu_e}(z_e)$ restricted to the above-mentioned variations. Hence, this is sufficient to determine formal stability of $[z_e]$ in the reduced Hamiltonian system. Let $\mathcal{S} \subset T_{z_e}T^*Q$ be the subspace of these variations, then one can choose $\mathcal{S}$ to be

$$\mathcal{S} = \underbrace{T_{z_e}\mathbf{J}^{-1}(\mu_e)}_{\ker D\mathbf{J}(z_e)} \cap \{\delta z \in T_{z_e}T^*Q \mid \langle\langle T\pi_Q \delta z, \, \eta_Q(z_e)\rangle\rangle_g = 0 \quad \forall \eta \in \mathcal{G}_{\mu_e}\},$$

where $\pi_Q : T^*Q \to Q$. But every point in $\mathbf{J}^{-1}(\mu_e)$ can be written as $p + \alpha_{\mu_e}(q)$, where $(q, p) \in \mathbf{J}^{-1}(0)$, equivalent to saying that $\langle p, \, \eta_Q(q)\rangle = 0$ for all $\eta \in \mathcal{G}$. Hence, $\mathcal{S}$ can be split in the following manner:

$$\mathcal{S} = \{\text{vert}_{z_e}(\delta p) \mid \langle \delta p, \, \eta_Q(q_e)\rangle = 0 \quad \forall \eta \in \mathcal{G}\} \oplus T_{z_e}\alpha_{\mu_e}\mathcal{V} = \mathcal{S}^{\dagger}_{\text{int}} \oplus T_{z_e}\alpha_{\mu_e}\mathcal{V}$$

where $\mathcal{V} = \{\delta_q \mid \langle\langle \delta q, \, \eta_Q(q_e)\rangle\rangle_g = 0 \quad \forall \eta \in \mathcal{G}_{\mu_e}\}$. Moreover, in most cases, one can split $\mathcal{V} \subset T_{z_e}Q$ as in the following:

$$\mathcal{V} = \{\zeta_Q(q_e) \mid \zeta \in \mathcal{G}^{\perp}_{\mu_e}\} \oplus \{\delta q \in \mathcal{V} \mid \langle \eta, \, (D\mathbb{I}(q_e) \cdot \delta q)\xi_e\rangle = 0 \quad \forall \eta \in \mathcal{G}^{\perp}_{\mu_e}\} = \mathcal{V}_{\text{rig}} \oplus \mathcal{V}_{\text{int}}.$$

$\mathcal{V}_{\text{rig}}$ is the subspace of variations generated by the group action, transverse to $T_{z_e}G_{\mu_e} \cdot z_e$. In the Abelian case, where $G_{\mu_e} = G$, $\mathcal{V}_{\text{rig}}$ vanishes. $\mathcal{V}_{\text{int}}$ on the other hand is the space of variations where loosely speaking $\{ \langle\langle \eta_Q(q_e), (\xi_e)_Q(q_e)\rangle\rangle = 0 \}$ for any $\eta \in \mathcal{G}_{\mu_e}^{\perp}$ looks invariant.[1] This choice of $\mathcal{V}_{\text{int}}$ results in a block-diagonalized $\delta^2 H_{\mu_e}(z_e)$. We note that vectors in $\mathcal{V}_{\text{rig}}$ relate to local variations of the **group variables**, while vectors in $\mathcal{V}_{\text{int}}$ roughly correspond to local variations of the **internal variables**, of the reduced Hamiltonian system.

Thus, using the notations $\mathcal{S}_{\text{rig}} = T_{z_e}\alpha_{\mu_e}\,\mathcal{V}_{\text{rig}}$ and $\mathcal{S}_{\text{int}} = T_{z_e}\alpha_{\mu_e}\,\mathcal{V}_{\text{int}}$, we have the following decomposition of $\mathcal{S}$

$$\mathcal{S} \;=\; \mathcal{S}_{\text{reg}} \oplus \mathcal{S}_{\text{int}} \oplus \mathcal{S}_{\text{int}}^{\dagger} \;\simeq\; \mathcal{G}_{\mu_e}^{\perp} \oplus \mathcal{V}_{\text{int}} \oplus \mathcal{S}_{\text{int}}^{\dagger} \;\simeq\; T_{z_e}P_{\mu_e}.$$

Let Q be of dimension n, G of dimension m, and G_μ of constant dimension $m' \leq m$ for all $\mu \in \mathcal{G}^*$. Then the three components of $\mathcal{S}$ will be respectively of dimensions, $m - m'$, $n - m$ and $n - m$, adding to $2n - m - m'$ the dimension of P_{μ_e}. $m - m'$ is an even number since it is the dimension of the symplectic leaf.

Let δ_s represent variation restricted to $\mathcal{S}$. Then, using the above decomposition of $\mathcal{S}$, $\delta_s^2 H_{\mu_e}(z_e)$ in local coordinates block-diagonalizes as follows

$$\delta_s^2 H_{\mu_e}(z_e) = \begin{bmatrix} A_{\mu_e} & 0 & 0 \\ 0 & \delta^2_{\mathcal{V}_{\text{int}}} V_{\mu_e} & 0 \\ 0 & 0 & \delta^2_{\mathcal{S}^{\dagger}_{\text{int}}} K_{\mu_e} \end{bmatrix} (z_e) = \text{Diag}(A,\, \Sigma,\, M^{-1}), \tag{5}$$

where $A_\mu : \mathcal{G}_{\mu_e}^{\perp} \times \mathcal{G}_{\mu_e}^{\perp} \to \mathcal{R}$ is the symmetric **Arnold form** given by

$$A_{\mu_e}(z_e)(\zeta, \eta) = \langle \text{ad}^*_\zeta \mu_e,\; \mathbb{I}(q_e)^{-1}\text{ad}^*_\eta \mu_e + \text{ad}_\eta \mathbb{I}(q_e)^{-1}\mu_e \rangle.$$

Now, $\delta^2_{\mathcal{S}^{\dagger}_{\text{int}}} K_{\mu_e}(z_e) = \delta^2_{\mathcal{S}^{\dagger}_{\text{int}}} \langle\langle p,\, p\rangle\rangle_g (= M^{-1})$ is clearly positive definite, thus to establish formal stability, it remains to show that $A_\mu(z_e)$ and $\delta^2_{\mathcal{V}_{\text{int}}} V_{\mu_e}(z_e)$ are also positive definite.

Similarly, one can obtain a representation of $\Omega(z_e)$ restricted to $\mathcal{S}$ in local coordinates, which is equivalent to a local representation of $\Omega_{\mu_e}([z_e])$ in P_{μ_e}. By using the facts that

$$\Omega(z)(\delta_1 z, \delta_2 z) = -\,d\alpha_{\mathbf{J}(z)}(\delta_1 q, \delta_2 q) \qquad \forall (\delta_1 z, \delta_2 z) = T_z\alpha_{\mathbf{J}(z)}(\delta_1 q, \delta_2 q), \tag{6}$$

and

$$\Omega(z)\,(T_z\alpha_\mu \eta_Q(q),\, T_z\alpha_\mu \zeta_Q(q)) = -\langle \mu,\, [\eta, \zeta] \rangle \qquad \forall \eta, \zeta \in \mathcal{G}, \quad \mu = \mathbf{J}(z), \tag{7}$$

one can show that in local coordinates, restriction of $\Omega(z_e)$ to $\mathcal{S}$ is of the form

$$\Omega_{\mathcal{S}}(z_e) = \begin{bmatrix} L & C & 0 \\ -C^T & -B & I \\ 0 & -I & 0 \end{bmatrix}, \qquad (\Omega_{\mathcal{S}}^{-1})^T = \begin{bmatrix} -L^{-1} & 0 & L^{-1}C \\ 0 & 0 & I \\ -C^T L^{-1} & -I & -S - C^T L^{-1} C \end{bmatrix}, \tag{8}$$

[1] Alternatively, we could have split $\mathcal{V}$ into $\mathcal{V}_{\text{rig}} \oplus \{ \delta q \mid \langle\langle \delta q, \eta_Q(q) \rangle\rangle_g = 0 \quad \forall \eta \in \mathcal{G} \}$.

where L is a skew-symmetric matrix corresponding to the Lie-Poisson bracket (7) on the coadjoint orbit of μ_e, C is a coupling term between internal and group variables, and B is the skew-symmetric magnetic term corresponding to $d\alpha_{\mu_e}$ in (6). For more details regarding the above derivations see Marsden (1992) and Simo, Lewis and Marsden (1991).

Therefore, instead of considering the linearized reduced Hamiltonian dynamics in local coordinates; *i.e.*

$$\dot{z} = X_{h_{\mu_e}}(z) \qquad \Longrightarrow \qquad \dot{\delta z} = \left(\Omega_{\mu_e}^{-1}([z_e])\right)^T \delta^2 h_{\mu_e}([z_e])\delta z, \tag{9}$$

the energy-momentum method considers the following equivalent linearized system, where $\delta_S^2 H_{\mu_e}(z_e)$ and $\Omega_S(z_e)$ are given by (5) and (8):

$$\dot{\overline{\delta}z} = \left(\Omega_S^{-1}(z_e)\right)^T \delta_S^2 H_{\mu_e}(z_e)\overline{\delta}z. \tag{9'}$$

Then, z_e will be formally and spectrally stable if $\delta_S^2 H_{\mu_e}(z_e)$ is positive definite.

Comparison of energy-Casimir and energy-momentum methods.
In the energy-Casimir method, one searches for Casimir functions on T^*Q/G, such that z_e dropped to the quotient is a critical point of $\delta(\overline{H} + C)$ and $\delta^2(\overline{H} + C)$ at that point is definite. $\overline{H}$ here denotes the Hamiltonian dropped to the quotient, *i.e.* $H = \overline{H} \circ \pi_G$, where $\pi_G : T^*Q \to T^*Q/G$.

A <u>sufficient</u> condition for existence of such Casimir is to find a function ϕ on $\mathcal{G}^*$ such that $\phi \circ \mathbf{J} = C \circ \pi_G$, ϕ is Ad*-invariant, and $\frac{\delta\phi}{\delta\mu}(\mu_e) = -\xi_e$. Since $\mathbf{J}$ is equivariant, Ad*-invariance of ϕ implies that $\phi \circ \mathbf{J}$ is G-invariant and hence, C is well-defined.

$$\begin{array}{ccccc} & & T^*Q & & \\ & \swarrow^{\pi_G} & & \searrow^{\mathbf{J}} & \\ T^*Q/G & & & & \mathcal{G}^* \\ & \searrow^{C} & & \swarrow^{\phi} & \\ & & \mathcal{R} & & \end{array}$$

Then, it is clear from the Noether's theorem that $C \circ \pi_G$ is conserved along the trajectories of every G-invariant Hamiltonian system on T^*Q, which is equivalent to saying that C is a Casimir. To see that $[z_e] \in T^*Q/G$ is a critical point of $\delta\,(\overline{H} + C)$, notice that

$$X_{\phi\circ\mathbf{J}}(z) = \left(\frac{\delta\phi}{\delta\mu}\Big|_{\mu=\mathbf{J}(z)}\right)_{T^*Q}.$$

Therefore at z_e, $X_{C\circ\pi_G}(z_e) = X_{\phi\circ\mathbf{J}}(z_e) = -\left(\xi_e\right)_{T^*Q}$, while by definition, $X_H(z_e) = (\xi_e)_{T^*Q}$. This means that z_e is a critical point of $X_{H+C\circ\pi_G} = X_H + X_{C\circ\pi_G}$ and thus a critical point of $\delta\,(H + C \circ \pi_G)$ and $\delta\,(\overline{H} + C)$.

Hence to apply the energy-Casimir method, it is sufficient to find ϕ satisfying the above properties. Note that Ad*-invariance of ϕ means that $\mathrm{ad}^*_{\frac{\delta\phi}{\delta\mu}}\mu = 0$, *i.e.* ϕ is a Casimir function with respect to the Lie-Poisson bracket on $\mathcal{G}^*$.

However, existence of such function ϕ or such Casimir function C is difficult to verify and may not hold. On the other hand, with the energy-momentum method one directly examines the second variation of the reduced Hamiltonian in $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, thus determining the stability of the equilibrium in the reduced structure or the relative equilibrium in the original system. Nevertheless, one should remember that the energy-momentum method results in stronger conditions for formal stability, that is to say there are situations where the energy-momentum method fails to prove stability, while there exist Casimir functions using which one can show that the system is formally stable.

3. Rayleigh Dissipation

In this section, we give an abstract definition of dissipative force field for Hamiltonian systems on cotangent bundles. In Bloch *et al* (1994b), the following derivation has been carried out for the Lagrangian representation of the system and the Euler-Lagrange equations. However, for the hyperregular case, one can directly derive similar definitions by using the Hamiltonian structure, as presented in the following:

Let X_d be a vector field on T^*Q. Then, we say this vector field is **vertical** if there exists a fiber-preserving map $F_d : T^*Q \to T^*Q$ such that

$$\Omega(X_d, w) = -\langle F_d, T\pi_Q w\rangle \qquad \forall w \in TT^*Q,$$

where $\pi_Q : T^*Q \to Q$. Then, we have

$$\begin{aligned} dH \cdot X_d(q,p) &= \Omega(X_H, X_d)(q,p) = \langle F_d, T\pi_Q X_H\rangle(q,p) \\ &= \langle F_d, \mathbb{F}L^{-1}\rangle(q,p) = \langle \widetilde{F}_d(q,v), v\rangle \qquad \text{for} \qquad (q,v) = \mathbb{F}L^{-1}(q,p) \end{aligned}$$

where we have used the fact that by definition of H and $\mathbb{F}L$, $T\pi_Q X_H = \mathbb{F}L^{-1}$. Therefore, if z_t is the integral curve of $X_H + X_d$, we have

$$\dot{H}(z_t) = \underbrace{dH \cdot X_H(z_t)}_{0} + \langle \widetilde{F}_d(v_q(t)), v_q(t)\rangle \qquad \text{where} \qquad v_q(t) = \mathbb{F}L^{-1}(z_t).$$

We call the map $\widetilde{F}_d$ (or F_d) a **force field.** If in addition $\langle \widetilde{F}_d(q,v), v\rangle \leq 0$ for all $v \in TQ$ such that $\dot{H}(z_t) < 0$, then we call $\widetilde{F}_d$ a **dissipative force field.** For such $\widetilde{F}_d$, if $\widetilde{F}_d(q,v) = -\mathbb{F}R(q,v) = -D_2R(q,v)$ for some function $R : TQ \to \mathcal{R}$, then R is called a **Rayleigh dissipation function.** In particular, for $R(q,v) = \langle\!\langle v, v\rangle\!\rangle_{\Gamma(q)}$, we have

$$\dot{H}(z_t) = -\mathbb{F}R(v_q(t)) \cdot v_q(t) = -\langle\!\langle v, v\rangle\!\rangle_{\Gamma(q_t)} < 0.$$

The above definition of force field, coincides with the standard definition, since one can show that the Lagrangian representation of $\dot{z} = X_H + X_d$ is given by

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L(q,\dot{q})}{\partial q} = \widetilde{F}(q,\dot{q}).$$

Now, let $\widetilde{F}_d = \mathbb{F}R$ be G-equivariant, which can be implied by G-invariance of R. Then, by equivariance of $\mathbb{F}L$, X_d will be Φ_g-invariant, *i.e.* $X_d(\Phi_g z) = T_z\Phi_g X_h(z)$, and group symmetry will be preserved for the perturbed Hamiltonian system $\dot{z} = X_H(z) + X_d(z)$.

But in addition, we would like the integral curves of $X_H + X_d$ to preserve the momentum or its corresponding coadjoint orbit. This means that we would like the sets $\mathbf{J}^{-1}(\mu)$ or $\mathbf{J}^{-1}(\mathcal{O})$ be preserved for any $\mu \in \mathcal{G}^*$ or $\mathcal{O}$ a coadjoint orbit in $\mathcal{G}^*$. In the first case, we must have

$$\begin{aligned} \frac{d}{dt}J^{\xi}(z) = \underbrace{dJ^{\xi}\cdot X_H(z)}_{0} + dJ^{\xi}\cdot X_d(z) &= -\Omega(X_d, \xi_{T^*Q})(z) \\ &= \langle F_d(z), \xi_Q(q)\rangle = 0 \qquad \forall \xi \in \mathcal{G}, \end{aligned} \tag{10}$$

where $\widetilde{F}_d : TQ \to T^*Q$ and we have used the fact that $T\pi_Q \circ \xi_{T^*Q} = \xi_Q \circ \pi_Q$. This imposes a strong restriction on $\widetilde{F}_d$. In particular if action of G on T^*Q is transitive, for example if $T^*Q = T^*G$, we would get $\widetilde{F}_d = 0$, *i.e.* there exists no force fields preserving the momentum map. Thus, it is more reasonable to look for dissipative force fields which preserve $\mathbf{J}^{-1}(\mathcal{O})$. Now, every member of the coadjoint orbit passing through $\mathbf{J}(z)$ can be represented as $\mathrm{Ad}_g^*\mathbf{J}(z)$ for some $g \in G$. Therefore, $\mathbf{J}^{-1}(\mathcal{O})$ is preserved by the integral curves of $X_H + X_d$ if

$$\frac{d\mathbf{J}(z)}{dt} = \mathrm{ad}^*_{\overline{\eta}(z)}\mathbf{J}(z) \qquad \Longleftrightarrow \qquad \begin{aligned} \frac{d}{dt}J^{\xi}(z) &= J^{[\overline{\eta}(z),\xi]}(z) \\ &= \langle F_d(z), \xi_Q(q)\rangle \qquad \forall \xi \in \mathcal{G}^*, \end{aligned} \tag{11}$$

for some $\overline{\eta} : T^*Q \to \mathcal{G}$. In §5, we use the above condition to explicitly formulate the coadjoint-orbit preserving force fields on T^*G and show that any such dissipative force field can be represented by a double bracket.

4. Dissipation Induced Instability: Dissipation Affecting the Internal Variables

Consider the Hamiltonian system described in §2 and the corresponding reduced dynamics. Let the Rayleigh dissipation function R be defined as

$$R(q,v) = \frac{1}{2}\langle\langle v^{\perp}, v^{\perp}\rangle\rangle_{\Gamma(q)},$$

where $v^{\perp}$ is the projection of v onto $\{\delta q \mid \langle\langle \delta q, \eta_Q(q)\rangle\rangle_g = 0 \ \ \forall \eta \in \mathcal{G}\}$, and Γ is positive definite G-invariant. Then, it is evident that R is G-invariant. This means that the corresponding force field $\widetilde{F}_d$ and vector field on T^*Q, X_d, are G-equivariant and Φ_g-invariant. Moreover, one can easily see in local coordinates that $\langle \widetilde{F}_d(v_q), \xi_Q(q)\rangle = 0$ for all $\xi \in \mathcal{G}$. Therefore, we know from (10) that $\mathbf{J}$ is preserved along the integral curves of $X_H + X_d$.

Hence, we can reduce the dynamics of the perturbed Hamiltonian system to P_{μ_e} corresponding to the relative equilibrium z_e. Using the definition of R, we have $R(q_c, v) = 0$ for $v \in \mathcal{V}_{\text{rig}}$. This means that the dissipation affects only the internal variables.

Thus in local coordinates, restriction of X_d to S in a neighborhood of z_e is given by

$$\Omega_S(z)\, X_d \Big|_S (z) = \begin{bmatrix} 0 \\ \Gamma(q)\, \mathbb{F}L^{-1}(q,\, p - \alpha_{\mu_e}(q)) \\ 0 \end{bmatrix}. \tag{12}$$

By using (8) and (9'), we then get the following linearized system at z_e:

$$\dot{\delta z} = \left(\left(\Omega_S^{-1}(z_e)\right)^T \delta_s^2 H_{\mu_e}(z_e) - \begin{bmatrix} 0 \\ 0 \\ \Gamma M^{-1} \end{bmatrix} \right) \delta z. \tag{13}$$

One can use (5) and (8) to expand the above equations and show that in fact for the linearized Hamiltonian

$$H^o(\delta r, \delta q, \delta p) = \frac{1}{2}\left(\delta p \cdot M^{-1} \delta p \,+\, \delta q \cdot \Sigma \delta q \,+\, \delta r \cdot A \delta r\right) \qquad \text{where} \qquad \delta z = (\delta r, \delta q, \delta p), \tag{14}$$

the dissipation term results in $\dot{H}(\delta z_t) \leq 0$.

Using the linearized equation (13), we get the following result:

Theorem 1. *Suppose A_{μ_e} is nondegenerate or vanishes altoghether and Γ corresponding to the dissipation vector field (12) is positive definite.*

Then if A and Σ defined in (5) are positive definite, z_e remains a nonlinearly and spectrally stable relative equilibrium of the perturbed Hamiltonian dynamics $\dot{z} = (X_H + X_d)(z)$, where X_d is given by (12).

On the other hand, if A or Σ has at least one negative eigenvalue and C in (8) has full row-rank, z_e will be Lyapunov and spectrally unstable for small dissipation of the given form affecting only the internal variables. □

The first statement immediately follows from (14). For the second case, as in example 1, we can find a perturbation of H^o in (14), W_{H^o}, such that $\dot{W}_{H^o}(\delta z_t) < 0$. See Bloch *et al* (1994a) for details. To show that z_e is also spectrally unstable we use the same argument given in §1.

We also remark that since C is $m - m' \times n - m$, the condition of the theorem, restricts the dissipation-induced instability result to the cases where $n + m' - 2m \geq 0$, where $n = \dim Q$, $m = \dim G$, $m' = \dim G_\mu$. This of course puts a lower bound on the number of internal variables for which this result can be used.

In addition, one can show that C has full row-rank, if $\mathcal{V}_{\text{int}}^{\perp} \cap T_{q_e}(G \cdot q_e) = \{0\}$.

5. Dissipation Induced Instability: Dissipation Affecting the Group Variables

In the last section, we studied the effect of dissipative force fields on stability of relative equilibria, restricting to the cases where the force field affects only the internal variables. In this section, we consider the case where the force field affects the group variables as well. In such cases one can not usually expect the momentum to be preserved. However, one can find force fields which preserve the coadjoint orbit through the momentum, and hence can carry on the reduction procedure with respect to $P_\mu \simeq \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$.

But first we classify the coadjoint-orbit preserving force fields on $T^*Q = T^*G$ and show that such force fields can be represented by a double bracket and be regarded as the gradient of a Rayleigh dissipation function tangent to symplectic leaves or coadjoint orbits.

5.1 Double-Bracket Force Fields for Lie-Poisson Equations

Example 2. We begin with the simple example of rigid body. The rigid body equations in $so(3)^*$ are given by

$$\dot{\Pi} = \Pi \times \Omega, \qquad \text{where} \qquad \Pi \in so(3)^*, \qquad \Omega = \mathbb{I}^{-1}\Pi \in so(3) \tag{15}$$

and $H(\Pi) = \frac{1}{2} M \cdot \Omega$ is the Hamiltonian. Let these equations be purturbed by the following term:

$$\dot{\Pi} = \Pi \times \Omega + \epsilon \Pi \times (\Pi \times \Omega) \qquad \text{for some} \quad \epsilon > 0. \tag{16}$$

Then, it is easy to see that $\dot{H} = -\epsilon(\Pi \times \Omega)^2 < 0$ around the equilibrium points and $\frac{d}{dt}\|\Pi\|^2 = 0$. Therefore, the added dissipation term preserves the coadjoint orbit $\{\|\Pi\| = \text{constant}\}$ while dissipating energy. Using H as Lyapunov function, then it is easy to show that the stability of the equilibria of (15) remains unchanged with addition of the above dissipation term. Notice that

$$\frac{\delta H}{\delta \Pi} \cdot (\Pi \times \Psi) = \Omega \cdot (\Pi \times \Psi) = -\frac{1}{\|\Pi\|^2}(\Pi \times (\Pi \times \Omega)) \cdot \Psi \qquad \forall \Psi \in \mathcal{G},$$

where every tangent vector to $\mathcal{O}_{\Pi_0} = \{\|\Pi\|^2 = \|\Pi_0\|^2\}$ can be represented by $\Pi \times \Psi$ for some $\Psi \in \mathcal{G}$. Thus, defining

$$\langle\langle \operatorname{Grad} H, \Psi \rangle\rangle_g = \frac{\delta H}{\delta \Pi} \cdot \Psi \qquad \forall \Psi \in \mathcal{G},$$

the above equalities mean that $-\frac{1}{\|\Pi\|^2}(\Pi \times (\Pi \times \Omega)) = \operatorname{Grad}_\Pi H$; the gradient of H tangent to $\mathcal{O}_\Pi$.

We now generalize this to the Lie-Poisson equations $\dot{\mu} = \operatorname{ad}^*_{\frac{\delta H}{\delta \mu}} \mu$ on the Poisson manifold $\mathcal{G}^*$ with Hamiltonian $h(\mu)$ and Poisson bracket $\{F, K\} = -\langle \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu}] \rangle$:

By (11), the coadjoint-orbit preserving force fields on T^*G satisfy $\langle F_d(g,\mu), \xi_G(g)\rangle = \langle \mathbf{J}(g,\mu), \mathrm{ad}_{\overline{\eta}(g,\mu)}\xi\rangle$ for some $\overline{\eta} : T^*G \to \mathcal{G}$. But for the left action on T^*G, $\mathbf{J}(g,\mu) = \mathrm{Ad}^*_{g^{-1}}\mu$ and $\xi_G(g) = T_e\mathcal{R}_g\xi$, where $\mathcal{R}_g$, $\mathcal{L}_g$ are the right and left multiplication by g. Thus, we have

$$\langle T_e^*\mathcal{R}F_d(g,\mu), \xi\rangle = \langle \mu, [\mathrm{Ad}_{g^{-1}}\overline{\eta}, \mathrm{Ad}_{g^{-1}}\xi]\rangle = \langle \mathrm{Ad}^*_{g^{-1}}\mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}\overline{\eta}}\mu, \xi\rangle$$

Then, using the facts that F_d is G-equivariant, *i.e.* $F_d(T_e^*\mathcal{L}_g\mu) = T_g^*\mathcal{L}_{g^{-1}}F_d(\mu)$, and $\mathrm{Ad}^*_{g^{-1}} = T_e^*\mathcal{R}_g \circ T_g^*\mathcal{L}_{g^{-1}}$, we find out that

$$F_d(\mu) = \mathrm{ad}^*_\eta\mu \qquad \text{where} \qquad \eta(\mu) = \mathrm{Ad}_{g^{-1}}\overline{\eta}(g,\mu). \tag{17}$$

Next, let us assume that the force field given by the above equality is the negative gradient of a function $\phi : \mathcal{G}^* \to \mathcal{R}$ tangent to the coadjoint orbit $\mathcal{O}_\mu$. Clearly, every element of $T_\mu\mathcal{O}_\mu$ can be represented by $\mathrm{ad}^*_\zeta\mu$ for some $\zeta \in \mathcal{G}$. Let $\langle\langle\cdot,\cdot\rangle\rangle_\Gamma$ be a metric on $\mathcal{G}^*$.
This metric induces a Γ^{-1}-metric on $\mathcal{G}$ and a *normal* metric on $T_\mu\mathcal{O}_\mu$ such that

$$\langle\langle \mathrm{ad}^*_\xi\mu, \mathrm{ad}^*_\zeta\mu\rangle\rangle_N = \beta\langle\langle\xi, \zeta\rangle\rangle_{\Gamma^{-1}} \qquad \text{for all } \xi,\zeta \in \mathcal{G} \text{ and some } \beta > 0.$$

Then we may define the gradient of ϕ tangent to $\mathcal{O}_\mu$ at μ by

$$\langle\langle \mathrm{Grad}_\mu\,\phi(\mu), \mathrm{ad}^*_\xi\mu\rangle\rangle = \langle \mathrm{ad}^*_\xi\mu, \frac{\delta\phi}{\delta\mu}\rangle \qquad \Longrightarrow \qquad \langle\langle \mathrm{ad}^*_{\eta(\mu)}\mu, \mathrm{ad}^*_\xi\mu\rangle\rangle_N = \langle \mathrm{ad}^*_{\frac{\delta\phi}{\delta\mu}}\mu, \xi\rangle,$$

which in turn implies

$$\eta(\mu) = \frac{1}{\beta}\Gamma\left(\mathrm{ad}^*_{\frac{\delta\phi}{\delta\mu}}\mu\right) \qquad \Longrightarrow \qquad -\mathrm{Grad}_\mu\,\phi = \mathrm{ad}^*_\eta\mu = \frac{1}{\beta}\mathrm{ad}^*_{\Gamma(\mathrm{ad}^*_{\delta\phi/\delta\mu}\mu)}\mu, \tag{18}$$

where $\Gamma : \mathcal{G}^* \to \mathcal{G}$ is induced by the metric. Thus, any coadjoint-orbit preserving force field which can be regarded as a gradient tangent to the coadjoint orbit is of the above form, where β in general can be replaced by any positive Casimir function on $\mathcal{G}^*$. Then, the time-derivative of the Hamiltonian $h(\mu)$ along the trajectories of

$$\dot{\mu}_t = \mathrm{ad}^*_{\frac{\delta h}{\delta\mu}}\mu_t + \frac{1}{\beta}\mathrm{ad}^*_{\Gamma(\mathrm{ad}^*_{\delta\phi/\delta\mu}\mu_t)}\mu_t \tag{19}$$

will be given by

$$\dot{h}(\mu_t) = \underbrace{\mathrm{ad}^*_{\frac{\delta h}{\delta\mu}}\mu_t\frac{\delta h}{\delta\mu}}_{0} + \mathrm{ad}^*_{\eta(\mu)}\mu_t\frac{\delta h}{\delta\mu} = -\frac{1}{\beta}\langle\langle \mathrm{ad}^*_{\frac{\delta h}{\delta\mu}}\mu_t, \mathrm{ad}^*_{\frac{\delta\phi}{\delta\mu}}\mu_t\rangle\rangle_\Gamma.$$

If for some ϕ, the above derivative is negative around equilibria, the corresponding force field is dissipative. One such case is when $\phi = h$.

Moreover, if the group G is compact and semisimple, then one can define a similar *normal* metric on adjoint orbits in $\mathcal{G}$, thus identifying $\mathcal{G}$ and $\mathcal{G}^*$. Assuming that Γ is identity, then the above orbit-preserving force field can be represented by the following **double bracket** form

$$-\text{Grad}_\mu \phi = \frac{1}{\beta}[\mu, [\mu, \frac{\delta\phi}{\delta\mu}]] \qquad \Longrightarrow \qquad \dot{\mu} = [\mu, \frac{\delta h}{\delta\mu}] + \frac{1}{\beta}[\mu, [\mu, \frac{\delta\phi}{\delta\mu}]]. \tag{20}$$

See Brockett (1993) for more details.

Finally, we should remark that although physical interpretation of dissipation fields affecting the internal variables is obvious, it is less obvious what kind of dissipations can be modeled as above. In Bloch *et al* (1994b), physical examples involving fluid mechanics and particle physics have been mentioned.

5.2 Instability Results

Consider the perturbed Lie-Poisson equations (19) with $\phi = h$, then for every $f : \mathcal{G}^* \to \mathcal{R}$,

$$\dot{f}(\mu_t) = \langle \mu_t, [\nabla h, \nabla f] \rangle - \frac{1}{\beta}\langle\langle \text{ad}^*_{\nabla h}\mu_t, \text{ad}^*_{\nabla f}\mu_t \rangle\rangle_\Gamma = \{f, h\}_{\text{Poisson}} - \{f, h\}_{\text{Pos.Sym.}}.$$

Then by using h as the Lyapunov function, we have the following result:

Theorem 2 *Let $\delta^2 h(z_e)$ be positive definite for z_e, an equilibrium of the unperturbed Lie-Posson equation. Then z_e is also a Lyapunov and spectrally stable equilibrium of the perturbed Lie-Posson equation (19) with $\phi = h$.*

On the other hand, if $\delta^2 h(z_e)$ is indefinite or negative semidefinite, addition of the dissipation term ensures that z_e is Lyapunov and spectrally unstable for the perturbed system (19) with $\phi = h$. □

To prove spectral instability in the final statement, note that since $\dot{h} < 0$, the linearization of h around z_e still implies Lyapunov instability in a local neighborhood of z_e. The rest follows as before.

Now consider the Hamiltonian structure on T^*Q introduced in §2 and studied in §4, now perturbed by a general coadjoint-orbit preserving dissipative force field satisfying (11); $\dot{z} = X_H + X_d$. In Bloch *et al* (1994b), the authors "implicitly" claim that in general, addition of such force field results in the following equivalent linearized system with respect to the reduced space of variations $\mathcal{S}$. Note that since $\mathbf{J}^{-1}(\mathcal{O})$ is preserved, we can still reduce the perturbed Hamiltonian system to $P_{\mu_e} \simeq \mathbf{J}^{-1}(\mathcal{O}_{\mu_e})/G$ and use the same decomposition of $\mathcal{S}$ for the variations of the linearized system. In this case, X_d in local coordinates and restricted to $\mathcal{S}$ in a neighborhood of z_e is given by

$$\Omega_{\mathcal{S}}(z) X_d \Big|_{\mathcal{S}} (z) = \begin{bmatrix} \Delta(q) \\ \Gamma(q)\, \mathbb{F}L^{-1}(q, p - \alpha_{\mu_e}(q)) \\ 0 \end{bmatrix}, \tag{21}$$

where $\Delta(q)$ is independent of internal variables. Then by using (8) and (9'), we get the following linearized system at z_e:

$$\dot{\delta z} = \left(\left(\Omega_S^{-1}(z_e)\right)^T \delta_s^2 H_{\mu_e}(z_e) - \begin{bmatrix} G^{-1}A \\ 0 \\ \Gamma M^{-1} \end{bmatrix} \right) \delta z. \tag{22}$$

If G is positive definite and Γ is positive semidefinite, we can again show that the Hamiltonian H^o given in (14) satisfies $\dot{H}(\delta z_t) \leq 0$. We should mention that G corresponds to the normal metric defined on the coadjoint orbits as explained in §5.1 and thus may be assumed to be positive definite. Then, we have the following result similar to theorem 1:

Theorem 3. *Suppose A_{μ_e} is nondegenerate, G is positive definite and Γ is positive semidefinite.*

Then if A and Σ are positive definite, z_e remains a nonlinearly and spectrally stable relative equilibrium of the perturbed Hamiltonian dynamics $\dot{z} = (X_H + X_d)(z)$, where X_d is given by (21).

On the other hand, if A or Σ has at least one negative eigenvalue and $C^TC + \Gamma$ is positive definite, z_e will be Lyapunov and spectrally unstable for small dissipation of the given form affecting the group and possibly the internal variables. □

Proof of the first statement is again immediate. The second statement follows from existence of another perturbation of H^o, $\overline{W}_{H^o}$, such that $\dot{\overline{W}}_{H^o}(\delta z_t) < 0$ and for small perturbations $\delta^2 \overline{W}_{H^o}$ is indefinite. Note that here condition of the theorem is satisfied when C has full column-rank; *i.e.* $m - m' \geq n - m$, while in theorem 1, C was required to have full row-rank. Therefore, if Γ is for example zero, the instability result is true if the number of internal variables is less than some upper bound.

6. The Heavy Top Example

In this section, we apply the above discussions to the example of heavy top.

Example 2. For the heavy top, the configuration space is $SO(3)$, the configuration of the body coordinates with respect to spatial coordinates, where the symmetry group is obviously the S^1 group of rotations around the direction of gravity.

Clearly, the symmetry group is an Abelian group acting freely and properly on $SO(3)$ and the Hamiltonian dynamics can be reduced to $T^*(SO(3)/S^1)$ of dimension 4. If in addition, the top is a **Lagrange top**, *i.e.* the top is symmetric with respect to its axis of rotation, the symmetry group will be the $S^1 \times S^1$ group of rotations around the gravity axis and top axis. The reduced dynamics on $T^*(SO(3)/(S^1 \times S^1))$ will be of dimension 2.

However, instead of this approach, one may consider the direction of the gravity as auxiliary variables, in which case the Hamiltonian will be invariant with respect to the action of the Euclidian group $SE(3)$, with the phase space being $T^*SE(3)$. Then, one can use Lie-Poisson reduction, to arrive at reduced dynamics on $se(3)$. We briefly review both approaches here, where we apply the results of §4 in the first case, and consider the double bracket dissipation term of §5, equation (21), in the second case.

I. Let Π, Ω, γ and $\mathbf{M} = \mathrm{M}\mathcal{E}_3$ be the angular momentum, the angular velocity of the top, the direction of gravity ($\|\gamma\| = 1$) and the coordinates of the center of mass ($\dot{M} = 0$) all given in body coordinates. Then the Hamiltonian or energy function is given by

$$H(\Pi, \gamma) = \frac{1}{2}\Pi \cdot \Omega \; + \; g\,\gamma.\mathbf{M}, \qquad \text{where} \qquad \Pi = \mathbb{I}\,\Omega \tag{23}$$

and $\mathbb{I}$ is the moment of inertia matrix with respect to the body coordinates. The system equations in spatial coordinates represent the unreduced dynamics. Let $\widehat{\Omega} := \Omega\times$. Then points in the phase space $T^*SO(3)$ can be represented by $(\Lambda, \Lambda\widehat{\Pi})$ where $\Lambda \in SO(3)$ is the configuration of the top in spatial coordinates and $\widehat{\Pi} \in so(3)$. The left action of $SO(3)$ on $T^*SO(3)$ will simply be $\Lambda_1 \cdot (\Lambda, \Lambda\widehat{\Pi}) = (\Lambda_1\Lambda, \Lambda_1\Lambda\widehat{\Pi})$. The Hamiltonian H in spatial coordinates, *i.e.* as a function on $T^*SO(3)$, is equal to

$$H(\Lambda, \Lambda\widehat{\Pi}) = \frac{1}{2}\Pi \cdot \mathbb{I}^{-1}\Pi \; + \; g\, e_3.\Lambda\mathbf{M}. \tag{24}$$

Clearly, H is not invariant with respect to the left action of $SO(3)$. However, if we assume the direction of gravity, e_3, to be an auxiliary variable, then one can assume $T^*SE(3)$ to be the phase space with elements represented by $(\Lambda, v, \Lambda\widehat{\Pi}, a)$, where $v, a \in \mathcal{R}^3$. The left action of $SE(3)$ on $T^*SE(3)$ is given by $(\Lambda_1, v_1)\cdot(\Lambda, v, \Lambda\widehat{\Pi}, a) = (\Lambda_1\Lambda, v_1(\Lambda_1 v), \Lambda_1\Lambda\widehat{\Pi}, v_1(\Lambda_1 a))$ where $v(a)$ means translation of a by v.

In this case, the Hamiltonian H as a function on $T^*SE(3)$ is given by the same equation as in (24). Using the facts that $\Lambda^T\Lambda = I$ for all $\Lambda \in SO(3)$ and the dynamics are translation-invariant, one can easily check that H is $SE(3)$-invariant. Thus, one may use Lie-Poisson reduction to reduce the dynamics to $se(3)$, with phase-space representation $(\widehat{\Pi}, \gamma)$. The reduced Hamiltonian is given by the original equation (23) in body coordinates and the Lie-Poisson equations are

$$\begin{pmatrix} \frac{d}{dt}\widehat{\Pi} \\ \frac{d}{dt}\gamma \end{pmatrix} = \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \begin{pmatrix} \widehat{\Pi} \\ \gamma \end{pmatrix} \qquad \text{where} \qquad \mu = (\widehat{\Pi},\, \gamma) \qquad \text{and} \qquad \frac{\delta H}{\delta \mu} = \left(\widehat{\Omega}, g\mathbf{M}\right).$$

But the ad* operation for $SE(3)$ which is the semiproduct of $SO(3)$ and $\mathcal{R}^3$ is given by

$$\mathrm{ad}^*_{(\widehat{v}, w)}(\widehat{u}, z) = (u \times v + z \times w,\; z \times v). \tag{25}$$

See Marsden, Ratiu and Weinstein (1984) for detailed analysis of Lie-Poisson reduction on semi-product groups. Thus, the Lie-Poisson equations can be written as

$$\dot{\Pi} = \Pi \times \Omega \; + \; g\gamma \times \mathbf{M}, \qquad\qquad \dot{\gamma} = \gamma \times \Omega. \tag{26}$$

The coadjoint orbits of $se(3)$ are given by $\{(\widehat{\Pi}, \gamma) \mid \Pi \cdot \gamma = \text{constant}, \;\; \|\gamma\| = \text{constant}\}$, which are 4-dimensional submanifolds of $\mathcal{R}^6$. In fact, one can check that along the trajectories of (26), $\frac{d}{dt}\Pi \cdot \gamma = 0$ and $\frac{d}{dt}\|\gamma\|^2 = 0$.

Now let us consider the double bracket dissipation term of equation (20) introduced in §5.2 with $\phi = H$. Using (25), one finds that the perturbed Lie-Poisson equations with the mentioned double bracket dissipation term are given by

$$\begin{cases} \dot{\Pi} = \Pi \times \Omega + g\gamma \times \mathbf{M} + \frac{1}{\beta}\left(\Pi \times (\Pi \times \Omega + g\gamma \times \mathbf{M}) + \gamma \times (\gamma \times \Omega)\right) \\ \dot{\gamma} = \gamma \times \Omega + \frac{1}{\beta}\left(\gamma \times (\Pi \times \Omega + g\gamma \times \mathbf{M})\right). \end{cases} \tag{27}$$

Now, by using reduction with respect to the S^1 symmetry group, Lewis *et al* (1992) show that in general, there exist formally unstable relative equilibria z_e, where $\delta^2 h_{\mu_e}([z_e])$ is indefinite. These relative equilibria reduce to equilibria of (26) or (27) and the reduced symplectic space is a coadjoint orbit of $se(3)$. Therefore, the conditions of theorem 2 are satisfied, meaning that these equilibria are nonlinearly and spectrally unstable for the perturbed system (27). The stable equilibria of the original system (26) remain stable.
We should however mention that the physical interpretation of the above dissipative term is unclear.

II. Next, let us consider reduction with respect to the S^1 symmetry group of rotations around the gravity axis. We first assume the general case where the top may be asymmetric. For this case, the symmetry group is Abelian and hence the reduced linearized system will be of the form given in (1) or (3).
Let Λ again represent the configuration of body with respect to the spatial coordinates and consider the dynamics in these coordinates, *i.e.* on $T^*SO(3)$, with phase space variable representation $(\Lambda, \widehat{\pi}\Lambda)$, where π is the angular momentum in spatial coordinates. The Hamiltonian with respect to these coordinates is then given by

$$H(\Lambda, \widehat{\pi}\Lambda) = \frac{1}{2}\underbrace{\pi \cdot \mathbb{I}_{\Lambda}^{-1}\pi}_{\Pi \cdot \mathbb{I}^{-1}\Pi} + g\, e_3 \cdot \Lambda \mathbf{M}, \qquad \text{where} \qquad \mathbb{I}_{\Lambda} = \Lambda \mathbb{I} \Lambda^T. \tag{28}$$

It is obvious that the left action of the symmetry group is the clockwise rotation (multiplication from left) around the gravity axis. The corresponding momentum map is given by the angular momentum along e_3. Therefore, the angular momentum in this direction is constant. The relative equilibria are the periodic trajectories which are invariant with respect to rotation around e_3.

Also noting that $\xi_{SO(3)}(\Lambda) = \xi\,\widehat{e_3}\Lambda$ for all $\xi \in \mathcal{R}^3$ it is easy to check that the locked inertia tensor is equal to $\mathbb{I}(\Lambda) = e_3 \cdot \mathbb{I}_{\Lambda} e_3$ constant along the trajectories of the relative equilibria. Let μ_e be the constant e_3-component of the angular momentum corresponding to the relative equilibrium z_e, then $\xi_e = \mathbb{I}(\Lambda_e)^{-1}\mu_e$ will be "the angular velocity of the axis of the top" at the relative equilibrium. Note that this is different from angular velocity of the top along e_3.

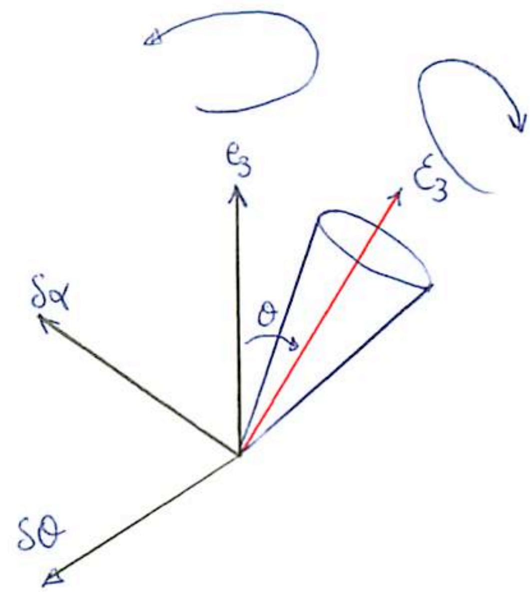

Fig. 1 The Heavy Top

To reduce the dynamics to $T^*(SO(3)/S^1)$ at z_e, we restrict Λ variations to $\mathcal{V} = \{\delta\Lambda \mid \delta\Lambda \perp e_3\}$ which is the space of angular velocities along the plane tangent to the direction of gravity. We choose $\mathcal{E}_3 \times e_3$ and $(\mathcal{E}_3 \times e_3) \times e_3$ as coordinates of this plane, where $\mathcal{E}_3$ is the unit vector along the rotation axis of the top.

Let θ be the tilt angle along the falling direction, then the components of $\mathcal{V}$ along $\mathcal{E}_3 \times e_3$ are variations in θ. Thus, variations of Λ in $\mathcal{V}$ coordinates can be represented by $(\delta\alpha, \delta\theta)$. See also figure 1. As explained in §2, to show that z_e is stable it suffices to show that $\delta^2 V_{\mu_e}(\Lambda_e)$ is positive definite along this directions, where V_{μ_e} for this example is given by

$$V_{\mu_e}(\Lambda) = g\, e_3 \cdot \Lambda\mathbf{M} \,+\, \frac{1}{2}\mathbb{I}(\Lambda)^{-1}\mu_e^2.$$

We refer to Lewis *et al* (1992) for the computations. There it is shown that in general for certain relative equilibria, $\delta^2 V_{\mu_e}(\Lambda_e)$ along $\mathcal{V}$ directions is indefinite.
The reduced linearized equations will be in terms of $(\delta\alpha, \delta\theta, \pi_1, \pi_2)$ where π_1, π_2 are the momenta along the $(\mathcal{E}_3 \times e_3) \times e_3$ and $\mathcal{E}_3 \times e_3$ axes. Let us now introduce dissipation terms of the form discussed in §4. For example we can chose the corresponding dissipative force field to be the negative gradient of the following Rayleigh dissipation function:

$$R(\Lambda, \delta\alpha, \delta\theta) = \frac{1}{2}(\sigma_1 \delta\alpha^2 + \sigma_2 \delta\theta^2).$$

Then the terms added to the linearized equations will be $-\sigma_1 \dot{\delta\alpha}$ for the π_1 equation and $-\sigma_1 \dot{\delta\theta}$ for the π_2 equation. These can be interpreted as friction against tilting sideways $(\delta\alpha)$ and tilting in the falling direction $(\delta\theta)$. Since we have assumed that the angular momentum along e_3 is preserved, there must however be no friction against rotations around e_3.

Thus according to theorem 1, the above friction destabilizes those relative equilibria for which $\delta^2 V_{\mu_e}(\Lambda_e)$ is indefinite ($\delta^2 V_{\mu_e}(\Lambda_e)$ along e_3 is always zero), while other relative equilibria remain stable.

Now, consider the case of Lagrange top. Using the principal body coordinates, $\mathbb{I} = \mathrm{Diag}(I_1, I_1, I_3)$ in this case, where I_3 is the moment of inertia along $\mathcal{E}_3$. Looking at (28), it is easy to see that H is invariant with respect to rotations around both the e_3 and $\mathcal{E}_3$ axes. The symmetry group is $S^1 \times S^1$, which is again Abelian, and the left action is defined as the clockwise rotation (multiplication from left) around the gravity axis and counterclockwise rotation (multiplication from right) around the top axis. The momentum map in this case is given by the angular momenta along e_3 and $\mathcal{E}_3$ which are constant. The infinitesimal generators are of the form $(\xi,\omega)_{SO(3)}(\Lambda) = \xi\,\widehat{e_3}\Lambda - \omega\Lambda\widehat{\mathcal{E}_3}$ and the locked inertia tensor is

$$\mathbb{I}(\Lambda) = \begin{bmatrix} e_3 \cdot \mathbb{I}_\Lambda e_3 & -e_3 \cdot \Lambda\mathbb{I}\mathcal{E}_3 \\ -e_3 \cdot \Lambda\mathbb{I}\mathcal{E}_3 & \mathcal{E}_3 \cdot \mathbb{I}\mathcal{E}_3 \end{bmatrix} = \begin{bmatrix} I_1 \sin^2\theta + I_3 \cos^2\theta & -I_3\cos\theta \\ -I_3\cos\theta & I_3 \end{bmatrix},$$

where θ is the tilt angle (see figure 1). Clearly, θ must be constant along the trajectories of relative equilibria. If $\theta \neq 0, \pi$, the action of the symmetry group on $SO(3)$ is free and proper and $\mathbb{I}(\Lambda)$ is invertible. We assume that $\theta \neq 0, \pi$. Then, $(\xi_e, -\omega_e) = \mathbb{I}(\Lambda_e)^{-1}\mu_e$ are respectively the angular velocity of the axis of the top, and the angular velocity of the top with respect to its frame, at the relative equilibrium z_e.

$\mathcal{V}$ in this case is only one-dimensional and is the space of $\delta\theta$ variations orthogonal to e_3 and $\mathcal{E}_3$. $\delta^2 V_{\mu_e}(\Lambda_e)$ is a real number which turns out to be positive for all $\theta \neq 0, \pi$. See Lewis *et al* (1992) for details. The only case where formal instability may occur is when $\theta = 0$, where for angular velocities less than a threshold the relative equilibrium becomes formally unstable.

Hence, adding dissipation terms of the form $-\sigma\,\dot{\delta\theta}$ to the reduced dynamics in this case results in nonlinear instability of the relative equilibrium. This form of dissipation can be interpreted as friction against tilting. Again we must assume that there is no friction against rotation. The other relative equilibria remain nonlinearly stable.

At the end, we note that the mentioned dissipative force fields are negative gradients of Rayleigh dissipation functions which are tangent to P_{μ_e}, a coadjoint orbit of $se(3)$ and preserve the coadjoint orbits. In §5.1, we showed that for the Lie-Poisson case, every such force field must be of the double bracket form (20) for some function ϕ. Thus one must be able to formulate these dissipation terms as double brackets added to the Lie-Poisson equations (26). However, the function ϕ in this case is not the Hamiltonian H. This means that for the heavy top the two classes of dissipative force fields are equivalent.

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The following notations and definitions are used in the following report:

Notations and Definitions:

- $P = T^*Q$: phase space, Ω : Canonical symplectic form on T^*Q
- G : symmetry group acting on Q with Lie algebra $\mathcal{G}$.
- Φ_g : (Left) Action of G on T^*Q.
- $\langle\cdot,\,\cdot\rangle : T^*Q \times TQ$ or $\mathcal{G}^* \times \mathcal{G} \to \mathcal{R}$.
- $\mathbb{F}: \mathcal{F}(TQ) \to T^*Q$: Fiber derivative; $\mathbb{F}R(q,v) = D_2R(q,v)$.
- $\langle\langle\cdot,\,\cdot\rangle\rangle_g$: g-Metric on TQ (T^*Q); $\langle\langle v_q,\, w_q\rangle\rangle_g := \langle \mathbb{F}L(v_q),\, w_q\rangle$, $\langle\langle p,\, p'\rangle\rangle_g(q) = \langle p,\, \mathbb{F}L^{-1}(q,p')\rangle$.
- δW, $\delta^2 W$: First and second variations of W; $\delta^2 W \equiv D^2 W$, $\delta W \equiv DW$.
- $L : TQ \to \mathcal{R}$: Lagrangian.
- $H : T^*Q \to \mathcal{R}$: Hamiltonian; $H(\mathbb{F}L(v_q)) := \|v_q\|_g^2 - L(q,v)$, $H(q,p) = K(q,p) + V(q)$, $K(q,p) := \frac{1}{2}\|p\|_g^2(q)$.
- X_H : Hamiltonian vector field corresponding to H.
- $z_e = (q_e, p_e)$: Ar relative equilibrium of the Hamiltonian system, $\mu_e = \mathbf{J}(q_e, p_e)$, $\xi_e = \mathbb{I}^{-1}(q_e)\mu_e$.
- $\eta_Q(q)$: Infinitesimal generator of G on Q $\forall \eta \in \mathcal{G}$.
- $\mathcal{O}_\mu$: Coadjoint orbit through μ; $\mathcal{O}_\mu := \{\,\mathrm{Ad}_g^*\mu,\ g \in G\,\}$, $T_\mu\mathcal{O}_\mu := \{\,\mathrm{ad}_\xi^*\mu,\ \xi \in \mathcal{G}\,\}$.
- G_μ, $\mathcal{G}_\mu$: Isotropy subgroup and subalgebra of $\mu \in \mathcal{G}^*$ w.r.t. the coadjoint action.
- $\mathcal{G}_\mu^\perp$: Complement of $\mathcal{G}_\mu$ with respect to $\mathbb{I}(q)$ at given q.
- $\mathbf{J} : T^*Q \to \mathcal{G}^*$: Momentum map; $\langle \mathbf{J}(q,p),\, \xi\rangle = \langle p,\, \xi_Q(q)\rangle$.
- $J^\xi(q,p) := \langle \mathbf{J}(q,p),\, \xi\rangle$ $\forall \xi \in \mathcal{G}$.

- $\mathbb{I}(q): \mathcal{G} \to \mathcal{G}^*:$ Locked Inertia Tensor; $\langle \mathbb{I}(q)\eta, \xi \rangle = \langle\langle \eta_Q(q), \xi_Q(q) \rangle\rangle_g$.

- $\varrho(q): T_qQ \to \mathcal{G}:$ Mechanical connection; $\varrho(q)(v_q); = \mathbb{I}^{-1}\mathbf{J}(\mathbb{F}L(v_q)) \quad \forall v_q \in T_qQ$.

- $\alpha_\mu(q) := \langle \varrho(q), \mu \rangle \qquad \forall \nu \in \mathcal{G}^*$.

- $H_\mu:$ Augmented (energy-momentum) Hamiltonian; $H_\mu := H - \langle \mathbf{J} - \mu, \xi \rangle$,
$\mu \in \mathcal{G}^*, \qquad \xi = \mathbb{I}^{-1}\mu$.

- $K_\mu, V_\mu:$ Amended Kinetic and potential energy; $H_\mu = K_\mu + V_\mu$,
$K_\mu := K(q, p - \alpha_\mu(q)), \qquad V_\mu(q) := V(q) + \frac{1}{2}\langle \mu, \xi \rangle, \qquad \xi = \mathbb{I}^{-1}\mu$.

- $K_\xi, V_\xi:$ Augmented Kinetic and potential energy; $H_\mu = K_\xi + V_\xi + \langle \mu, \xi \rangle$,
$K_\xi := K_\mu, \qquad V_\xi := V_\mu - \langle \mu, \xi \rangle$.

- $h_\mu([q],[p]) := K([q],[p]) + V_\mu([q]) = H_\mu([q],[p + \alpha_\mu]):$
Reduced Hamiltonian on $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$.

- $\Omega_\mu:$ Symplectic form on $T^*(Q/G_\mu)$ corresponding to the reduced Hamiltonian.

We note that using the above definitions, we have

$$\alpha_\mu(q) = \mathbb{F}L(\xi_Q(q)) \qquad \text{if} \qquad \mu = \mathbb{I}\,\xi.$$