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Good!

In this paper, we will study the existence of "internal resonances" which may arise for free oscillations of systems having nonlinearities.

To illustrate more precisely what is meant by this, we consider "cherry's equations", which come from the hamiltonian

$$H = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) + \frac{1}{2} p_2 (p_1^2 - q_1^2) - q_1 q_2 p_1$$

The linearization of the resulting equations of motion corresponds to 2 decoupled SHO's, and hence the linearized solutions are 'marginally' stable. [Here, 'marginally' means that the eigenvalues for the linearized system have zero real part].

However, it can be shown that the full solution to Cherry's eqn's are

? — just a 1-parameter family of sol'ns

$$q_1 = -\sqrt{2}\,\frac{\cos(t-T)}{t-T} \qquad p_1 = \sqrt{2}\,\frac{\sin(t-T)}{t-T}$$

$$q_2 = \frac{\cos 2(t-T)}{t-T} \qquad p_2 = \frac{\sin 2(t-T)}{t-T}$$

Note that by choosing T large, we can make q_1, q_2, p_1, p_2 arbitrarily small. And yet in a finite amount of time, namely $t = T$, the solutions blow up. What has occurred is that the presence of nonlinearities in the eqns. destroyed the stability of the corresponding linear system. Such is what is meant by internal resonances.

In this paper we wish to understand how such internal resonances can arise. To do this, I will first do a rather simple perturbation scheme on cherry's eqn's which will illustrate the salient features of this system. Second, I will describe some of the more general methods and results which may be found in the literature.

Cherry's eqns - A simple perturbation scheme

Consider a slightly generalized version of Cherry's hamiltonian:

$$H = \frac{1}{2} w_a (q_a^2 + p_a^2) + \frac{1}{2} w_b (q_b^2 + p_b^2) + \frac{1}{2} p_b (p_a^2 - q_a^2) - q_a q_b p_a$$

The resulting equations are

$$\dot{q}_a = w_a p_a + p_b p_a - q_a q_b$$

$$\dot{p}_b = -w_a q_a + p_b q_a + q_b p_a$$

$$\dot{q}_b = w_b p_b + \frac{1}{2}(p_a^2 - q_a^2)$$

$$\dot{p}_b = -w_b q_b + q_a p_a$$

Now expand:

$$q_a = \epsilon q_{a1} + \epsilon^2 q_{a2} + \ldots$$

$$p_a = \epsilon p_{a1} + \epsilon^2 p_{a2} + \ldots$$

$$q_b = \epsilon q_{b1} + \epsilon^2 q_{b2} + \ldots$$

$$p_b = \epsilon p_{b1} + \epsilon^2 p_{b2} + \ldots$$

$\underline{O(\epsilon)}$:

$$\dot{q}_{a1} = w_a p_{a1} \qquad \dot{q}_{b1} = w_b p_{b1}$$

$$\dot{p}_{a1} = -w_a q_{a1} \qquad \dot{p}_{b1} = -w_b q_{b1}$$

$$\longrightarrow \begin{cases} q_{a1} = a e^{i w_a t} + \bar{a} e^{-i w_a t} \\ p_{a1} = i a e^{i w_a t} - i \bar{a} e^{-i w_a t} \end{cases}$$

$$\begin{cases} q_{b1} = b e^{i w_b t} + \bar{b} e^{-i w_b t} \\ p_{b1} = i b e^{i w_b t} - i \bar{b} e^{-i w_b t} \end{cases}$$

$\underline{O(\epsilon^2)}$:

$$\dot{q}_{a2} = w_a p_{a2} + p_{b1} p_{a1} - q_{a1} q_{b1}$$

$$\dot{p}_{a2} = -w_a q_{a2} + p_{b1} q_{a1} + q_{b1} p_{a1}$$

$$\dot{q}_{b2} = w_b p_{b2} + \tfrac{1}{2}(p_{a1}^2 - q_{a1}^2)$$

$$\dot{p}_{b2} = -w_b q_{b2} + q_{a1} p_{a1}$$

—Substituting $O(\epsilon)$ solutions yields

$$\begin{cases} \dot{q}_{a2} = w_a p_{a2} - [2ab\, e^{i(w_a+w_b)t} + c.c.] \\ \dot{p}_{a2} = -w_a q_{a2} - [2i a\bar{b}\, e^{i(w_a-w_b)t} + c.c.] \end{cases}$$

$$\begin{cases} \dot{q}_{b2} = w_b p_{b2} - [a^2 e^{2iw_a t} + c.c.] \\ \dot{p}_{b2} = -w_b q_{b2} + [i a^2 e^{2iw_a t} + c.c.] \end{cases}$$

where c.c. denotes complex conjugate

—To solve, rewrite as two second-order eqn's:

$$\ddot{q}_{a2} = -w_a^2 q_{a2} - \left[2ia\bar{b}\, w_a\, e^{i(w_a-w_b)t} + 2iab(w_a+w_b)\, e^{i(w_a+w_b)t} + c.c.\right]$$

$$\ddot{q}_{b2} = -w_b^2 q_{b2} + \left[ia^2(w_b - 2w_a)\, e^{2iw_a t} + c.c.\right]$$

To solve, must distinguish between 2 cases:

Case 1 : $w_b \pm 2w_a \neq 0$

Here we find that

$$q_{a2} = +2ia\bar{b}\,\frac{w_a}{w_b(w_b-2w_a)}\, e^{i(w_a-w_b)t} + \frac{-2iab\,(w_a+w_b)}{w_a^2 + (w_a+w_b)^2}\, e^{i(w_a+w_b)t}$$

$$+ \ c.c.$$

$$q_{b2} = \frac{ia^2}{(w_b+2w_a)}\, e^{2iw_a t} + c.c.$$

Hence we see that in this case the leading correction to the linearized solution is once again oscillatory.

We must note several things about this case:

1) For our perturbation scheme to be valid, we require $\epsilon q_{a1} >> \epsilon^2 q_{a2}$, $\epsilon q_{b1} >> \epsilon^2 q_{b2}$, etc.

   But by looking at the denominators in q_{a2}, q_{b2} , we see it is necessary to introduce a <u>detuning parameter</u> δ, defined by

   $$w_b \pm 2w_a = \delta$$

   Then we see that

   $$\epsilon q_1 \sim O(\epsilon)$$

   $$\epsilon^2 q_2 \sim O\left(\frac{\epsilon^2}{\delta}\right)$$

   Perturbation scheme valid for $\epsilon q_1 >> \epsilon^2 q_2$

   $$\Rightarrow \quad |\epsilon| >> |\epsilon^2/\delta| \quad \Rightarrow \quad \boxed{\delta >> \epsilon}$$

2) Nothing has been said about $O(\epsilon^3)$ and higher corrections to the solution

<u>Case 2</u>: $w_b + 2w_a = 0$

Here, the eqn's of motion become

$$\ddot{q}_{a2} = -w_a^2 q_{a2} - \left[2ia\bar{b}w_a e^{3iw_a t} - 2iabw_a e^{-iw_a t} + c.c.\right]$$

$$\ddot{q}_{b2} = -(2w_a)^2 q_{b2} - \left[4ia^2 w_a e^{2iw_a t} + c.c.\right]$$

This system corresponds to SHO's driven at their natural frequency $\rightarrow$ resonance!

More explicitly, we can solve these eqn's exactly, and show that the expressions for q_1, p_1, q_2, p_2 all contain <u>secular</u> terms of the form

$$q_1, p_1 \sim te^{-iw_a t} \quad ; \quad q_2, p_2 \sim te^{+2iw_a t}$$

Hence the oscillations grow in amplitude due to the presence of the nonlinear terms. This is called a "2-1 resonance" (I think). yes!

Note: Since perturbation scheme valid only for $\epsilon q_1 \gg \epsilon^2 q_2$, we need

$$\epsilon \gg \epsilon^2 t$$

Hence, our $O(\epsilon^2)$ corrections valid only for times $t \ll \frac{1}{\epsilon}$. Thus we can claim that the oscillations might increase in amplitude only on time scales $t \ll \frac{1}{\epsilon}$. We cannot say what will occur on longer time ~~scla~~ scales, e.g. whether or not the solutions will equilibrate, or continue to increase without bound.

Ideally, we would like to add arbitrary nonlinear terms to a free oscillator system, and deduce the stability. This proves very difficult in general. Instead, will show how resonances typically arise, and how they can be removed:

Canonical Perturbation Theory [Dynamics, S. Rasband]
[Regular + Stochastic Motion, Lichtenberg + Lieberman]

To show why resonances can frequently arise, view the nonlinear terms as perturbations to a system of free oscillators.
Separate the hamiltonian into 2 pieces: a free oscillator piece (completely integrable), and a nonlinear (perturbation) term.
In action-angle variables,

$$H(J,\theta) = H_0(J) + \epsilon H_1(J,\theta)$$

where J, θ = action/angle variables coming from unperturbed H_0. They are not action/angle variables for H, but they are still valid canonical coordinates

If $\epsilon \ll 1$, might expect H to be integrable. If so, then look for new action/angle variables J', θ' such that $H(J,\theta) = H(J')$. So find canonical transf. from J, θ to J', θ':

write generating function

explain $$S(J',\theta) = \theta^i J_i' + \epsilon S_1(J',\theta) + \ldots$$

Since S = type 2 generating function,

$$\theta'^i = \theta^i + \epsilon \frac{\partial S_1}{\partial J_i'} + \ldots \quad , \quad J_i = J_i' + \epsilon \frac{\partial S_1}{\partial \theta^i} + \ldots$$

H_1 is periodic in action/angle

$$\Rightarrow \quad H_1(\underline{J},\theta) = \sum_{\underline{m}} H_{1\underline{m}}(J)\, e^{i\underline{m}\cdot\underline{\theta}} \qquad \vec{m} = (m_1, m_2, \dots m_n)$$

m_i = integer

Expand S_1 :

$$S_1(\underline{J}',\theta) = \sum_{\underline{m}\neq 0} S_{1\underline{m}}(J')\, e^{i m\cdot\theta}$$

Now write H_0 in terms of $\underline{J}'$:

$$H_0(\underline{J}) = H_0(\underline{J}') + \epsilon \underbrace{\left.\frac{\partial H_0}{\partial J_j}\right|_{\underline{J}'}} \frac{\partial S_1}{\partial \theta^j}$$

replace by $\left.\dfrac{\partial H_0}{\partial J_j}\right|_{J} = w^j(J)$

$$\Rightarrow \quad H_0(\underline{J}) = H_0(\underline{J}') + \epsilon\, w^j(\underline{J}) \frac{\partial S_1}{\partial \theta^j}$$

From above, $\dfrac{\partial S_1}{\partial \theta^j} = \displaystyle\sum_{m\neq 0} i m_j\, S_{1\underline{m}}(J')\, e^{i m\cdot\theta}$

Hence,

$$H(\underline{J}') = H_0(\underline{J}') + \epsilon\left[\sum_{\underline{m}\neq 0} i w^j m_j\, S_{1\underline{m}}(\underline{J}')\, e^{i\underline{m}\cdot\theta} + \sum_{m} H_{1\underline{m}}(\underline{J})\, e^{i m\cdot\theta}\right] + \dots$$

Now set coefficient of each $e^{i m\cdot\theta}$ to zero!

$$\Rightarrow \quad i\, w(J)\cdot m\, S_{1m}(\underline{J}') + H_{1m}(\underline{J}) = 0 \qquad m \neq 0$$

$$\Rightarrow \quad S_{1m}(J') = \frac{i H_{1m}(\underline{J})}{w(J)\cdot m}$$

Hence, $$\boxed{S_1(\underline{J}',\theta) = \sum_{\underline{m}\neq 0} \frac{i H_{1m}(\underline{J})}{w(J)\cdot m}\, e^{i\underline{m}\cdot\theta}}$$

$\Rightarrow$ Get <u>Resonance</u> if $w(J)\cdot m = 0$

i.e. problem of small devisors!

Thus we see the problems that resonances can produce.

Next, we outline two methods for removing resonances. What is actually meant by this is that the resonances can be pushed back to higher order in the perturbation expansion!

Removal of Resonances: [Lichtenberg & Lieberman]

The basic idea is to eliminate the resonant variables from the unperturbed H by jumping into a frame rotating at the resonant frequency.

Hence, these new coordinates measure slow oscillations about their resonant variables. [Assume 2 degrees of freedom]

let $H = H_0(J) + \epsilon H_1(J, \Theta)$ as before

$$H_1 = \sum_{l,m} H_{l,m}(J) \exp(i\, n \cdot \Theta) \qquad n = (l, m)$$

$$\omega_1(J) = \frac{\partial H_0}{\partial J_1} \qquad \omega_2(J) = \frac{\partial H_0}{\partial J_2}$$

Assume a resonance exists:

$$\frac{\omega_2}{\omega_1} = \frac{r}{s}$$

Let F_2 (Generating Function) be given by

$$F_2 = (r\Theta_1 - s\Theta_2)\hat{J}_1 + \Theta_2 \hat{J}_2$$

The new variables $\hat{J}, \hat{\Theta}$ defined by

$$J_1 = \frac{\partial F_2}{\partial \Theta_1} = r\hat{J}_1$$

$$J_2 = \frac{\partial F_2}{\partial \Theta_2} = \hat{J}_2 - s\hat{J}_1$$

$$\hat{\Theta}_1 = \frac{\partial F_2}{\partial \hat{J}_1} = r\Theta_1 - s\Theta_2$$

$$\hat{\Theta}_2 = \frac{\partial F_2}{\partial \hat{J}_2} = \Theta_2$$

Hence we're now in a rotating frame

$$\hat{\dot{\theta}}_1 = r\dot{\theta}_1 - s\dot{\theta}_2 = \text{slow deviation from resonance}$$

but $\hat{H}(\hat{p},\hat{q},t) = H(p,q,t) + \frac{\partial}{\partial t} F_2(q,\hat{p},t)$

Hence,

$$\hat{H} = \hat{H}_0(\vec{J}) + \epsilon \hat{H}_1(\vec{J},\vec{\theta})$$

where $$\hat{H}_1 = \sum_{l,m} H_{lm}(\vec{J}) \exp\left[\frac{i}{r}\left[l\vec{\theta}_1 + (ls+mr)\vec{\theta}_2\right]\right]$$

Since $\hat{\theta}_2$ is the fast variable, can average over it:

$$\bar{H} = \bar{H}_0(\vec{J}) + \epsilon \bar{H}_1(\vec{J},\vec{\theta}_1)$$

where $\bar{H}_0 = \hat{H}_0(\vec{J})$

$$\bar{H}_1 = \langle \hat{H}_1(\vec{J},\vec{\theta})\rangle_{\hat{\theta}_2} = \sum_{p=0}^{\infty} H_{-rp,ps}(\vec{J}) \exp(-ip\vec{\theta}_1)$$

$$\Rightarrow \hat{J}_2 = \hat{J}_{20} = \text{constant}$$

$$\vec{J}_2 = J_2 + \frac{s}{r} J_1 = \text{const.}$$

Since $\vec{J}_2 = \text{const.}$, $\bar{H}$ above $\approx$ motion w/ single degree of freedom.

Hence, have pushed back resonance terms.

Lastly, we give a sketch of another procedure to remove resonances:

<u>Method of Multiple time Scales:</u> [Nayfeh : Nonlinear Oscillations]

Going back to the original example, we found that at a resonance, secular terms of the form $\sim t\exp(\omega t)$ appeared. Hence, the first order correction to the ~~un~~ linear solution is valid only for $t \ll \frac{1}{\epsilon}$.

There is a way to get corrections which are valid to

$$t \ll \frac{1}{\epsilon^m}$$

The basic idea is to introduce multiple time scales:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \dots + \epsilon^{m-1}\frac{\partial}{\partial T_{m-1}}$$

This increases the number of variables. By choosing them appropriately, secular terms can be removed.

It turns out that for systems having quadratic nonlinearities, the ~~linear behavior~~ behavior is the same as the linear system to second order provided that $w_n \not\approx 2w_m$ or $w_n \not\approx w_m \mp w_k$.

Here's an example of method of multiple time scales for 2 coupled oscillators. [Note: damping is present → nonhamiltonian]

$$\ddot{u}_1 + w_1^2 u_1 = -2\hat{\mu}_1 \dot{u}_1 + \alpha_1 u_1 u_2$$

$$\ddot{u}_2 + w_2^2 u_2 = -2\hat{\mu}_2 \dot{u}_2 + \alpha_2 u_1^2$$

let $u_1 = \epsilon u_{11}(T_0, T_1) + \epsilon^2 u_{12}(T_0, T_1) + \dots$

$u_2 = \epsilon u_{21}(T_0, T_1) + \epsilon^2 u_{22}(T_0, T_1) + \dots$

Assume $\hat{\mu}_j = \epsilon \mu_j$

$\underline{O(\epsilon)}$: 1) $\frac{\partial^2}{\partial T_0^2} u_{11} + w_1^2 u_{11} = 0$

2) $\frac{\partial^2}{\partial T_0^2} u_{21} + w_2^2 u_{21} = 0$

$\Rightarrow$ $u_{11} = A_1(T_1)\exp(iw_1 T_0) + \text{c.c.}$

$u_{21} = A_2(T_1)\exp(iw_2 T_0) + \text{c.c.}$

$\underline{O(\epsilon^2)}$: 1) $\frac{\partial^2}{\partial T_0^2} u_{12} + w_1^2 u_{12} = -2\frac{\partial}{\partial T_0}\left[\frac{\partial}{\partial T_1} u_{11} + \mu_1 u_{11}\right] + \alpha_1 u_{11} u_{21}$

$$= -2iw_1(A_1' + \mu_1 A_1)\exp(iw_1 T_0) + \alpha_2\{A_1 A_2 \exp[i(w_1+w_2)T_0] + A_2 \bar{A}_1 \exp[i(w_2 - w_1)T_0]\} + \text{c.c.}$$

2) $$\frac{\partial^2}{\partial T_0^2} u_{22} + w_2^2 u_{22} = -2 \frac{\partial}{\partial T_0}\left[\frac{\partial}{\partial T_1} u_{21} + \mu_2 u_{21}\right] + \alpha_2 u_{11}^2$$

$$= -2iw_2 (A_2' + \mu_2 A_2) \exp(iw_2 T_0) + \alpha_2 [A_1^2 \exp(2iw_1 T_0) + A_1 \bar{A}_1] + c.c.$$

Suppose we're near a resonance:

$$w_2 = 2w_1 + \epsilon\sigma$$

detuning parameter

$$\Rightarrow \quad 2w_1 T_0 = w_2 T_0 - \sigma T_1$$

$$(w_2 - w_1) T_0 = w_1 T_0 + \sigma T_1$$

To remove secular terms from $O(\epsilon^2)$ eqns, require

$$-2iw_1 (A_1' + \mu_1 A_1) + \alpha_1 A_2 \bar{A}_1 \exp(i\sigma T_1) = 0$$

$$-2iw_2 (A_2' + \mu_2 A_2) + \alpha_2 A_1^2 \exp(-i\sigma T_1) = 0$$

Now the idea is to solve these eqns. Hence, the correction to the linearized problem will be valid for

$$t \ll \frac{1}{\epsilon^2}$$

$\Rightarrow$ resonances pushed back