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CDS 205

CDS 205: Final Homework-*A Look at Suris' Approach to Integrable Discretizations*

As computers become a more vital part of our study of physics and engineering, it is natural to desire to bridge the gap between the well-developed tools of dynamical systems theory and numerical analysis. Since computerized calculation is basically a discrete process, clever means for discretizing dynamical systems are of great interest to the physicist or engineer. In doing such a discretization, one of the major qualitative aspects of a dynamical system that one might seek to preserve is that of integrability (i.e. if the continuous dynamical system has functionally independent integrals in involution (w.r.t. a given bracket), then so too should the discretized version of the same). The method of time discretization presented by Yuri B. Suris is one such method of integrability for a restricted class of Hamiltonian systems. Since Suris neglects to provide proofs for the steps in his method (at least in his review papers), a natural way to achieve mastery of the theory that he uses is to provide these proofs where they are missing, or to flesh out the few that are provided.

Consider a Poisson Manifold, P, with Poisson bracket {•,•}; H(u) be an integrable Hamiltonian: i.e. $u = \{H,u\}$. We are interested in the class of Hamiltonian systems which admit a *Lax Representation.* $\exists T: P \to g, A: P \to g$ (where g is a Lie Algebra) such that the above differential equation can be represented as $T = [T,A]$. In addition to admitting such a representation, we also restrict ourselves to the class of systems for which the map A admits an *R-operator* (i.e. A=R(f(T)) where R is a linear operator: g→g, and f is Ad-Covariant function on g: $AdU \quad f(T) = f(AdU \quad T)$. Here Ad is the adjoint representation of algebra g, a conjugation operation (like that seen in class when constructing the inner automorphisms for determining the Lie Bracket of an algebra). It is important to not that this limits the claim of universality of this approach greatly, as Suris states his belief that certainly many interesting systems admit such a representation, but probably not all.

Since the idea of an R-matrix is so important to proceeding to build the structure that is to follow, it is prudent to spend a few minutes taking a deeper look at these objects. Let g be a Lie Algebra, R a linear operator: g→g. We say that R is a classical R-matrix if the bracket: $[X,Y]_R = [RX,Y] + [X,RY]$ is a Lie Bracket (i.e. satisfies Poisson Identity as skew-bilinearity follows trivially from fact [•, •] is a Lie Bracket). In such a case we define (g,R) as a double lie algebra equipped with bracket $[.,.]_R$. To connect this Lie algebra point of view with the integrable systems, let us now consider on $C^\infty(g^*)$ corresponding Lie-Poisson brackets:

$$\{h_1,h_2\}_0(L) = L([X_1,X_2])$$
$$\{h_1,h_2\}_R(L) = L([X_1,X_2]_R) = L([RX_1,X_2] + [X_1,RX_2])$$
$$X_i = dh_i \in g; L \in g^*$$

Letting Ad*, and Ad_R* be the coadjoint representations of the corresponding algebras g, g_R. We note that from definition of the Lie Bracket that: $Ad_R^* X \quad L = Ad^* RX \quad L + R^* (Ad^* X \quad L)$

Using this observation we can make our first connection between the classical R-matrix and the linear Lie-Poisson structure:

Let(G,R) be a double Lie Algebra

i) Ad* invariant functions are in involution with respect to the Lie Poisson brackets on $C^\infty(g^*)$

ii) The Equations of motion on g* with an invariant Hamiltonian, h, with respect to Lie Poisson bracket $\{.,.\}_R$ can be written in two equivalent forms.

$$\frac{dL}{dt} = Ad_R^* dh(L) \quad L$$

$$\frac{dL}{dt} = Ad^* R(dh(L)) \quad L$$

Here the proof of both parts is primarily a verification exercise:

i) Let h_1,h_2 be Ad-invariant functions on g*:

$\{h_1,h_2\}_0(L) = L([X_1,X_2]) = Ad^* X_2 \quad L(X_1) = 0$ by dint of the fact that X_1,X_2 are derivatives of Ad-invariant functions.

$$\{h_1,h_2\}_R = L([RX_1,X_2]) + L([X_1,RX_2]) = L([RX_1,X_2]) - L([RX_2,X_1])$$
$$= Ad^* X_2 \quad L(RX_1) - Ad^* X_1 \quad L(RX_2) = 0$$

as both terms in the sum vanish due to the Ad-invariance property.

ii) Because h is an invariant Hamiltonian with respect to the R-bracket (Lie-Poisson), the equations of motion can be written as:

$$\frac{dL}{dt} = \{L,h\}_R = L([L,X]_R) = Ad_R^* X \quad L$$

But h is Ad-invariant, so let us use our observation from above:

$Ad_R^* X \quad L = Ad^* RX \quad L + R^*(AdX \quad L)$, applying Ad-invariance of h forces the second term to go to zero. This leaves the desired second form of the equations of motion.

$$\frac{dL}{dt} = \{L,h\}_R = Ad^* RX \quad L$$

So we have seen that the determination of R-brackets on the Poisson manifold and the Lie algebra can be a powerful tool in connecting the Lax representation to the integrable system. However, it remains to be seen, outside of an extremely lucky guess, how an appropriate R-operator is determined. In order to establish some sufficient conditions, let us go back to the definition that $[.,.]_R$ is a Lie bracket if and only if it satisfies the Jacobi Identity:

$$\left[X,[Y,Z]_R\right]_R = \left[RX,[Y,Z]_R\right] + \left[X,R[Y,Z]_R\right]$$
$$= \left[RX,[RY,Z] + [Y,RZ]\right] + \left[X,R[RY,Z] + R[Y,RZ]\right]$$
$$= \left[-X,[RY,RZ]\right] + \left[-X,[Y,Z]\right] + \left[X,[Y,Z]_R\right]$$

Adding in the cyclic permutations of the second term yields the Jacobi Identity for the standard Lie Bracket: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$, which leaves us with the cyclic permutations of the first and third terms ➔ $[.,.]_R$ satisfies the Jacobi Identity iff:

$$[X,[RY,RZ]-R([Y,Z]_R)]+[Y,[RZ,RX]-R([Z,X]_R)]+[Z,[RX,RY]-R([X,Y]_R)]=0$$

Two ways of forcing this condition to hold lead to the Yang-Baxter Sufficient conditions:

i) Force each second term in the brackets to be zero:

$$[RX,RY]=R([X,Y]_R)\forall X,Y\in g$$

This sufficient condition is known as the Yang-Baxter Equation.

ii) Force the terms to conform to the Jacobi Identity for the standard Lie Bracket:

$$[RX,RY]-R([X,Y]_R)=-[X,Y]\forall X,Y\in g$$

We note that by bilinearity of the Lie bracket, it is actually possible to place any non-zero constant in front of the RHS and not change the Jacobi Identity (i.e. $[RX,RY]-R([X,Y]_R)=-\alpha[X,Y]$). This leads to the family of sufficient conditions, known as Modified Yang-Baxter Equations (MYB(R;α)).

Together these results enable us to connect the Integrable structure on the Poisson manifold to the Lax representation on the Lie algebra. Taking an arbitrary smooth function, Ad-invariant function,φ, we can write the equations of motion (as shown above) for R satisfying MYB:

$$\frac{dL}{dt}=\{\phi,L\}_R=[T,R(\nabla\phi)]$$

But is this linear bracket construction satisfactory? It would seem, keeping one eye towards eventually achieving some sort of discretization, that a higher order bracket structure might be preferable. With this in mind, let us look towards obtaining a quadratic bracket that is compatible with $\{.,.\}_R$.

In order to get a higher order bracket, let us first imagine that the r-operator is a tensor: $R=R^{uv}e_u\otimes e_v\in g\otimes g$, then we know that the Bracket defined:

$\{\phi,\varphi\}=R^{uv}\left(\frac{\partial\phi}{\partial x^u}\frac{\partial\varphi}{\partial x^v}-\frac{\partial\phi}{\partial x^v}\frac{\partial\varphi}{\partial x^u}\right)$ satisfies the definition of a Poisson Bracket (i.e. will satisfy the Jacobi identity et al.) if R is skew. What we want to do is to connect this bracket to integrable systems. A more general way to approach the problem certainly is to recast this bracket in terms of the left and right actions of operators:

Let G be the Lie group corresponding to Lie algebra g, then for L in G:

$\rho_L,\lambda_L:g\to T_Lg$ are the derivatives of the left and right (resp.) actions of L

$\rho_L^*,\lambda_L^*:T_L^*g\to g^*$, their duals. Then we have the bracket defined above equivalent to:

$\{h_1,h_2\}(L)=\rho_L^*(X_2)R(\rho_L^*(X_1))-\lambda_L^*(X_2)R(\lambda_L^*(X_1))$, with X_i:=dh, as above.

$\rho_L^*(X)=LX,\lambda_L^*(X)=XL$

$\{h_1,h_2\}(L)=LX_2(R(LX_1))-X_2L(R(X_1L))$

If we further constrain the algebra of the matrix group to be associative ([X,Y]=XY-YX) and to have an invariant scalar product ($\langle (X \quad Y), Z\rangle = \langle X, (Y \quad Z)\rangle$). Then we can define a nice quadratic bracket:

$$\{h_1, h_2\}_2(L) = \langle L, [X_1, R(LX_2)] + [R(X_1L), X_2]\rangle.$$

So we wish to show three things about this bracket:

i) $\{.,.\}_2$ is skew and satisfies Jacobi Identity.

ii) $\{.,.\}_2$ and $\{.,.\}_R$ form a Hamiltonian Pair (linear combinations are still Poisson Brackets)

iii) Ad-invariant functions are in involution with respect to $\{.,.\}_2$

In terms of proving i) I am somewhat at a loss. Though I have been assured a proof exists by looking through the literature (especially Semenov-Tyan-Shanskii's) paper on R-matrix theory. However, he intimates that the proof is both somewhat tedious and needs to be approached in terms of the de Rham complex. Since the only familiarity I have with this is through Douglas Arnold's Colloquium on Differential Complexes in FEM, I do not think I am equipped to attempt this proof. Since this is not a major point of debate, or even really integral (NPI) to Suris' approach, I will have to be happy just taking his word for it.

For ii) let us just take some combination: $h^\lambda(L+\lambda I)=:h(L)\lambda$

$$\{h_1^\lambda, h_2^\lambda\}_2(L+\lambda I) = \langle (L+\lambda I), [X_1, R((L+\lambda I)X_2)] + [R(X_1(L+\lambda I)), X_2]\rangle$$
$$= \{h_1, h_2\}_2(L) + \lambda\{h_1, h_2\}_R$$

The proof of involution in iii) will be just like that of the proof for $\{.,.\}_R$:

$\frac{dL}{dt} = [L, R(XL)]$; from the lax form of the linear bracket. Then using the fact h is Ad-invariant, meaning

$$XL = LX \Rightarrow \{h_1, h_2\}_2(L) = < L, [X_1, R(LX_2)] + [R(X_1L), X_2] > = < L, [X_1, R(X_2L)] + [R(X_1L), X_2] >$$

But h_1, h_2 are ad-invariant, and thus already known to be in involution with respect to the linear bracket. So both terms in the inner product must vanish.

A better way in which to see this, and get a more friendly quadrization, is to take the bracket as defined and choose three operators, A_1,A_2 skew, and S some linear operator such that: $R = A_1 + S = A_2 + S^*$

Plugging this into the definition of the Quadratic bracket leaves:

$$\{h_1, h_2\}_2(L) = \langle A_1(X_1L), X_2L\rangle - \langle A_2(LX_1), LX_2\rangle + \langle S(LX_1), X_2L\rangle - \langle S^*(X_1L), LX_2\rangle$$

Before showing the involution property, it is important to recover the conditions on A_1,A_2,S for the bracket to be Poisson as defined. Fortunately this is fairly simple. We know that the bracket will be Poisson if R satisfies MYB. But, if two linear operators satisfy MYB, so does the sum of the two (follows from the bilinearity of the Lie bracket as well as that of the inner product. So a sufficient condition for R to satisfy MYB will simply be for A_1,A_2, and S to satisfy MYB.

The benefit of this way of writing the quadrization of the Bracket is when we look at the Hamiltonian system we get with some Ad-invariant function:

$$\frac{dL}{dt} = \{h,L\}_2 = L \quad A_1(XL) - A_2(LX) \quad L + L \quad S(LX) - S^*(XL) \quad L$$

Again, Ad-invariance means that XL=LX, but this implies that, grouping together like terms:

$$= L \quad (A_1 + S)(XL) - (A_2 + S^*)(XL) \quad L$$
$$= L \quad R(XL) - R(XL) \quad L$$
$$= [L, R(XL)]$$

Seeing that the two brackets yield the same Lax form for the equations of motion forces the conclusion that the functions in involution for one must be in involution for the other. So involution of Ad-invariant functions for the quadratic bracket follows easily.

Presumably it is possible to find brackets in this way for even higher orders (i.e. cubic brackets and beyond), by refining the requirements on the Lie algebra. It seems that for an extremely high order bracket one is trading in utility (if we keep refining eligible Lie algebras, the class of problems that can be solved by the revelation of increasingly clever brackets will grow smaller).

With that said, let us return from our excursion into quadrization and look at R-operators which arise from the simple operation of splitting the Lie group corresponding to the Lie algebra. Since these form the basis for the Suris' recipe for integrable discretization, they are worth understanding. Again we take g to be an associative Lie Algebra with non-degenerate, invariant scalar product. Because it is then a linear space, we know from operator theory that we can split it into two subalgebras, g_1, and g_2 such that: $g_1 \oplus g_2 = g$. Defining π_+,π_- to be corresponding projections into the two subalgebras, we can define an R-operator on this algebra as $R := \pi_+ - \pi_-$. We then need to verify that R indeed satisfies the MYB equation:

$$[(\pi_+ - \pi_-)X, (\pi_+ - \pi_-)Y] - (\pi_+ - \pi_-)([(\pi_+ - \pi_-)X, Y] + [X, (\pi_+ - \pi_-)Y])$$
$$[(\pi_+ - \pi_-)X, (\pi_+ - \pi_-)Y] - (\pi_+ - \pi_-)([(\pi_+ - \pi_-)X, (\pi_+ + \pi_-)Y] + [(\pi_+ + \pi_-)X, (\pi_+ - \pi_-)Y])$$

Letting X_+,X_-,etc. be the respective projections:

$$[(\pi_+ - \pi_-)X, (\pi_+ - \pi_-)Y] - (\pi_+ - \pi_-)([X_+, Y_+ + Y_-] - [X_-, Y_+ + Y_-] + [X_+, Y_+ - Y_-] + [X_-, Y_+ - Y_-])$$
$$= [X_+ - X_-, Y_+ - Y_-] - (\pi_+ - \pi_-)(2[X_+, Y_+] - 2[X_-, Y_-])$$
$$= [X_+ - X_-, Y_+ - Y_-] - 2[X_+, Y_+] - 2[X_-, Y_-]$$
$$= [X_+, -Y_+ - Y_-] + [X_-, -Y_+ - Y_-] = -[X, Y]$$

Which is the desired result. It is also interesting to remark that in the above verification we get for free the fact that (g,R) is a double algebra with: $[X,Y]_R = 2[X_+,Y_+] - 2[X_-,Y_-]$

Since the R as given is an R-operator for a linear bracket, we know then that we get a Lax representation for the equations of motion for Ad-covariant ($AdU \quad f(T) = f(AdU \quad T)$) functions f:g →G

$$\frac{dL}{dt} = [L, R(f(L))] = [L, \pi_+(f(L)) - \pi_-(f(L))]$$, which because of ad-covariance of f, becomes:

$$\frac{dL}{dt} = [L, \pi_+(f(L))] = -[L, \pi_-(f(L))].$$

The final step before applying this to differential equations is to define corresponding projections on Lie group G, having Lie algebra g. Letting G_+, G_- be the subgroups corresponding to subalgebras g_+,g_- we can define the group projection operators: Π_+, and Π_- such that for T in G, $T=\Pi_+(T)\Pi_-(T)$.

Finally we wish to attempt to solve the equations of motion given in Lax Form:

The solution to the Differential Equation given by: $\dfrac{dT}{dt} = [T,\pi_+(f(T))] = -[T,\pi_-(f(T))]$ with initial condition T_0 is given by:

$$T(t) = \Pi_+^{-1}(\exp(tf(T_0)))T_0\Pi_+(\exp(tf(T_0))) = \Pi_-(\exp(tf(T_0)))T_0\Pi_-^{-1}(\exp(tf(T_0)))$$

The proof is by a direct verification:

Let us define: $P = \Pi_+(\exp(tf(T_0)));M = \Pi_-(\exp(tf(T_0)))$, then plugging in:

$$T(t) = P^{-1}T_0P = MT_0M^{-1}$$

$$\frac{DT}{dt} = P^{-1}T_0 P - P^{-1} P P^{-1}T_0P = M T_0M^{-1} - MT_0M^{-1} M M^{-1}$$

$$= T \quad P^{-1} P - P^{-1} P \quad T = -T \quad M M^{-1} + M M^{-1} \quad T$$

Exploiting the fact that we have set ourselves up in an associative algebra:

$\dfrac{dT}{dt} = [T,P^{-1} P] = -[T,M M^{-1}]$, the Lax equations we are looking for. However it remains to be shown that the projections of f(T) are indeed the same as the two terms in the Lie Brackets.

$$\exp(tf(T_0)) = \Pi_+(\exp(tf(T_0)))\Pi_-(\exp(tf(T_0))) = P(t)M(t)$$

$$\exp(tf(T_0))f(T_0) = \dot{P} M + P \dot{M} = PMf(T_0)$$

which by Ad-covariance of $f(T_0)$:

$$= Pf(T)M = P M + P M$$

$$\Rightarrow f(T) = P^{-1} P + M M^{-1}$$

But, using the fact that the subalgebras are linear spaces, it is clear that the first term must lie in g_+, with the second term in g_-. By definition of direct sum, we know that the decomposition of f(T) into the two subalgebras is unique. So we conclude:

$\pi_+(f(T)) = P^{-1} P;\pi_-(f(T)) = M M^{-1}$. Which ensures that the solution given is a solution to the Lax equations, thus concluding the verification.

Now that the methodology of the proof has been placed in the open for the continuous system, the final step in this process is to apply it to a proposed solution to the discrete evolution equations for an integrable system:

Selecting some Ad-covariant function F:g →G, the solution to the difference equation (the discrete analogs of the Lax equations):

$T_{n+1} = \Pi_+^{-1}(F(T_n))T_n\Pi_+(F(T_n)) = \Pi_-(F(T_n))T_n\Pi_-^{-1}(F(T_n))$, with IC, T_0 is given by:

$$T_n = \Pi_+^{-1}(F^n(T_0))T_0\Pi_+(F^n(T_0)) = \Pi_-(F^n(T_0))T_0\Pi_-^{-1}(F^n(T_0))$$

The verification will be very similar, again defining $P_n = \Pi_+(F^n(T_0)); M_n = \Pi_-(F^n(T_0))$ and plugging into the difference equation:

$$\begin{aligned}
&F^n(T_0) = P_n M_n \\
&T_n = P_n^{-1}T_0P_n = M_nT_0M_n^{-1} \\
&\Rightarrow T_{n+1} = P_{n+1}^{-1}T_0P_{n+1} = M_{n+1}T_0M_{n+1}^{-1} \\
&= P_{n+1}^{-1}P_nP_n^{-1}T_0P_nP_n^{-1}P_{n+1} = M_{n+1}M_n^{-1}M_nT_0M_n^{-1}M_nM_{n+1}^{-1}
\end{aligned}$$

Defining then two new combinations: $L_n = P_n^{-1}P_{n+1}; U_n = M_{n+1}M_n^{-1}$, the difference equation becomes:

$T_{n+1} = L_n^{-1}T_nL_n = U_nT_nU_n^{-1}$ the form of the "Discrete Lax Equations". All that remains to show is that once again our combinations are indeed projections of the Ad—covariant functions:

$P_{n+1}M_{n+1} = F^{n+1}(T_0) = P_nM_nF(T_0)$, but just as last time, the covariance of the function allows us to "bring it inside"

$$\begin{aligned}
&P_{n+1}M_{n+1} = P_nF(T_n)M_n \\
&\Rightarrow P_n^{-1}P_{n+1}M_{n+1}M_n^{-1} = F(T_n) \\
&\Rightarrow L_nU_n = F(T_n)
\end{aligned}$$

Appealing again to the definition of direct sum concludes the verification.

So we can conclude this examination with Suris' "recipe" for an integrable discretization of an Hamiltonian system. If we are able to write our system in a way that admits a Lax representation:

$\dfrac{dL}{dt} = [L,\pi_+(f(L))] = -[L,\pi_-(f(L))]$ then we may simply write the difference equation in the form above with the function $F(L)=I+hf(L)+O(h^2)$ (a first order approximation to f, which will produce a first order discretization $x'=x+h\{H,x\}+O(h^2)$ where H is the Hamiltonian). Clearly from what we have seen above, this is not the only admissible recipe (i.e. it seems feasible to form difference Lax equations for any R operator that satisfies MYB). However, this approach is still a little disappointing. Though Suris examines in some detail in his papers Toda and Volterra lattices, it seems that even for these well-studied examples, it takes quite a bit of doing to set everything into the Lax form that is needed (seemingly) for his recipes. The approach seems nice if you have a good intuition (for example Suris works mostly with gl(N) in his examples) for which Poisson manifold and Lie algebra to work with in order to get a workable Lax form. Without this intuition it seems that this approach leaves you stumbling around in the dark without any hope of getting a discretization that can work. Basically telling you to pick an operator and hope it satisfies MYB. It left me curious to see if this could even be done for a lattice with a multi-well potential between particles or other quirky behavior. Though going through the proofs was fun for what it was, the paper itself seems to raise more questions than it answers by proffering a seemingly simple tool theoretically that looks nearly impossible to use in practice unless you already know the answer you are looking for.

Literature Used (in Descending Order of Importance):

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