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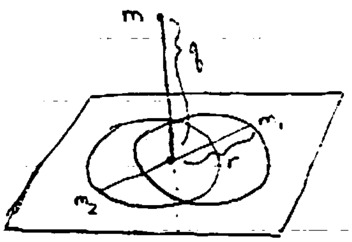
Twist Maps, Billiards, and Celestial Mechanics

References: K. Sitnikov, "The Existence of Oscillatory Motions in the Three-Body Problem", Soviet Physics Doklady, Vol. 5, No. 4, 1961, pp 647-650 (English) and J. Moser, "Stable and Random Motions in Dynamical Systems", Annals of Mathematical Studies No. 77, 1973.

1. Restricted 3-Body Problem

Imagine court astronomer of a double star solar system trying to predict the intervals between successive visits of a comet given previous data.

Ignore m, comet's mass, so that primaries execute 2-body motion.
Assume primary's masses are equal, and normalize unit so $m_1 = m_2 = \frac{1}{2}$
Normalize time so period of the primaries is 2π.
Normalize length so gravitational constant is 1.
Place origin at primary's center of mass.
Assume comet starts on a line perpendicular to primary's plane.



So $K.E. = \frac{1}{2} m \dot{q}^2 \qquad P.E. = -m(q^2+r^2)^{-\frac{1}{2}} \qquad p = m\dot{q}$

$$H(q,p,t) = \frac{p^2}{2m} - m\left(q^2 + r^2(t)\right)^{-\frac{1}{2}}$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q} = -mq\left(q^2+r^2(t)\right)^{-3/2}$$

$$\ddot{q} = -\frac{q}{(q^2+r^2(t))^{3/2}} \quad \text{where} \quad r(t) = r(t+2\pi) \quad \text{and}$$

$$r(t) = \frac{1}{2}(1-\varepsilon\cos t) + O(\varepsilon^2) \quad \text{with } \varepsilon = \text{eccentricity of elliptical orbits.}$$

Note: can think of setting $m = 1$

Suppose solution $q(t)$ has (a possibly infinite number of) zeroes at $\ldots t_{k-1} < t_k < t_{k+1} \ldots$.
Set $s_k = \left[\frac{t_{k+1} - t_k}{2\pi} \right] \in \mathbb{N}$ to be the integer that measure how many complete primary revolutions between comet visits.

Theorem: Given a sufficiently small $\varepsilon > 0$, there exists an integer $m(\varepsilon)$ such that any sequence $\{s_k\}_{k=-\infty}^{\infty}$ with $s_k \geq m(\varepsilon)$ corresponds to a solution $q(t)$ of this restricted three body problem

Remarks: Can admit half infinite sequences, and finite sequences ending with ∞.
Actually works for all $0 < \varepsilon < 1$ except a discrete set.
Can allow $m > 0$ as shown by Alekseev.
Shows can have unbounded solutions with infinitely many zeroes, many periodics, etc.

II. Twist Maps

Give an idea of the construction used in proving this
Orbit determined by giving time t_0 s.t. $q(t_0) = 0$ and giving $\dot{q}(t_0)$.
Notice invariance under $t_0 \mapsto t_0 + 2\pi$ and $\dot{q}(t_0) \mapsto -\dot{q}(t_0)$.
Therefore, can take t_0 and $v_0 \equiv |\dot{q}(t_0)|$ as polar coordinates on $\mathbb{R}^2$.

Define $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ by setting $\phi(t_0, v_0) = (t_1, v_1)$ if t_1 is the next time bigger than t_0 s.t. $q(t_1) = 0$ and if $v_1 = |\dot{q}(t_1)|$ where q is a solution with $q(t_0) = 0$ and $\dot{q}(t_0) = v_0$. Where is ϕ defined?

Consider case $\varepsilon = 0$, so $r = \frac{1}{2}$. Conservation of energy says (taking $m = 1$)

$$H(p,q) = \frac{p^2}{2} - \left(q^2 + \frac{1}{4}\right)^{-\frac{1}{2}} = \frac{\dot{q}^2}{2} - \left(q^2 + \frac{1}{4}\right)^{-\frac{1}{2}} = \text{constant}$$

For $H<0$, these are closed curves in $(q,\dot{q})$ plane. For $H \geq 0$, extend to ∞.
For $H<0$, curves intersect $q=0$ at $\dot{q}=v$ with

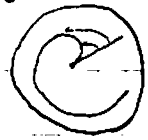
$$v^2 = 2\left(H + \left(\tfrac{1}{4}\right)^{-\frac{1}{2}}\right) = 2(H+2) < 4$$

So that ϕ is defined on the disk in $\mathbb{R}^2$ where $v<2$.

What does ϕ look like in this case? Conservation of energy gives $v_1 = v_0$.
Moreover, return time $t_1 - t_0 = T(v_0)$ is independent of t_0 and $T(v_0) \to \infty$ as $v_0 \to 2$.

$$\phi\begin{pmatrix} t_0 \\ v_0 \end{pmatrix} \equiv \begin{pmatrix} t_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} t_0 + T(v_0) \\ v_0 \end{pmatrix}$$

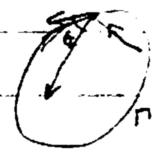
Poincaré Section



Notice that ϕ is an area preserving monotone twist map.
Both properties persist for $\varepsilon > 0$. Think of ϕ as canonical change of coordinates.
Twist means that the larger the initial velocity the longer before it comes back.

III. Billiards

Suggest at least why twist maps have lots of periodic orbits by considering this model problem.
Let C denote closed, convex, smooth, oriented curve in plane.
Code bounces by location and cosine of angle of reflection.



Phase space is annulus $C \times [-1,1]$.
Bounce map $\Psi: C\times[-1,1] \to C \times [-1,1]$ is area preserving.
Also monotone twist, since next bounce location is monotone in the angle of reflection.

__Theorem__: For each $\frac{p}{q} \in \mathbb{Q}$ there is a closed billiard trajectory of period q whose successive bounces have the same ordering on C as iterates of a rotation through angle $\frac{2\pi p}{q}$.

__Proof__: Maximize perimeter of inscribed q-gons, $\frac{p}{q}$-gons, etc. (Start with ellipse)
__Remark__: In fact, there are two, the other being a minimax (cf. Poincaré's Last Geometric Theorem)