DOMITILLA DEL VECCHIO

Very nice & clear report!

Introduction

In this report I describe the formulation of the Isoholonomic problem as seen from different points of view. It turns out that the so called cat's problem can be formulated as an Optimal Control problem and the resulting equations which drive the dynamics of the solution are the Wong's equations of a particle in a Yang-Mills field. Moreover these solutions can also be derived by reviewing the same problem as the one of finding sub-Riemannian geodesics. Once the problem has been already formulated as an Optimal Control one, considering sub-Riemannian metric, it transforms almost naturally in the problem of finding geodesics. In what follows I will first rewiew some concepts and definitions which will be used later.

Basic Concepts

Geodesics

Be Q a pseudo-Riemmanian manifold with metric $g_q =<,>_q$ at point q in Q. Be $L : TQ \to \mathbb{R}$:

$$L(v) = 1/2 < v, v >_q = 1/2 g_{ij} v^i v^j$$

The Lagrangian vector field will be $S : TQ \to T^2Q$:

$$S(q, v) = ((q, v), (v, \gamma(q, v)))$$

where the "acceleration term" γ is given by the Euler Lagrange equations:

$$\gamma^i(q, v) = -\Gamma^i_{jk}(q) v^j v^k \ ,$$

where

$$\Gamma^i_{jk}(q) = \frac{1}{2} g^{hl} \left(\frac{\partial g_{il}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right)$$

so that the geodesics are governed by:

$$\ddot{q} = \gamma(q, \dot{q}) = -\Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k \ .$$

The integral curves of S projected on Q are called the geodesics of the metric g. Note that if $Q = \mathbb{R}^3$ with g the standard inner product, then $\Gamma_{jk} = 0$ so that $\dot{q} = \mathit{constant}$ and the particle describes a stright trajectory that is the shortest path between two points. Note also that when the Lagrangian is a quadratic expression in the velocity the resulting trajectories are geodesics.

Covariant Derivative

$\nabla : X(Q) \times X(Q) \to X(Q)$ is defined as

no dot

$$\nabla_X Y(q) = -\gamma_q(X(q), Y(q)) + DY(q)\dot{X}(q)$$

and if γ is the one as defined above this is called the *Levi Civita Covariant Derivative.*

Be $c(t)$a curve in Q and $u = \dot{c}(t)$, then the covariant derivative of X along c is

$$\frac{DX}{Dt} = \nabla_u X$$

and by taking $\dot{c}(t) = X(t)$ if $\frac{DX}{Dt} = \nabla_{\dot{c}}\dot{c} = 0$ then $c(t)$ is a geodesic since

$$\ddot{c}(t) - \gamma(\dot{c}, \dot{c}) = 0 \ .$$

Here we have therefore another definition of geodesics:

a curve c(t) is a geodesic if $\nabla_{\dot{c}}\dot{c} = 0$.

A Particle in a Magnetic Field and Wong's equations

A charged particle in $\mathbb{R}^3$ mooving under the influence of a magnetic field has a dynamics given by the Lorentz law:

$$m\frac{dv}{dt} = \frac{e}{c}v \times B$$

where B is the magnetic field and $v = (\dot{x}, \dot{y}, \dot{z})$. These equations can be shown to be Hamiltonian if we choose the non standard symplectic 2-form in $\mathbb{R}^3$ $\Omega_B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) + \frac{e}{c}B$ with Hamiltonian function $H = \frac{m}{2}((\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2)$, so that X_H can be easily determined by

$$dH = i_{X_H}\Omega_B$$

.

This fact suggests that choosing a non canonical symplectic form we could have a Lagrangian function that is a quadratic expression in the velocities, then allowing geodesic solutions.Therefore consider the map $t_A : (x, v) \to (x, p)$ with $p = mv + \frac{e\mathbf{A}}{c}$, then Ω_B transforms to the standard 2-form and the hamiltonian becomes

$$H_A = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|$$

where $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \nabla \times \mathbf{A}$. In this way let the configuration space $Q_k = \mathbb{R}^3 \times S^1$ with variables (q, θ) with $A = A^\flat$ and consider $\omega = A + d\theta$ called the connection one form. Define the Kaluza-Klein Lagrangian as

$$L_k(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2}(m\|\dot{q}\|^2 + (\mathbf{A} \cdot \dot{q} + \dot{\theta})^2)$$

with momenta

$$\mathbf{p} = m\dot{q} + (\mathbf{A} \cdot \dot{q} + \dot{\theta})\mathbf{A}$$

$$p = (\mathbf{A} \cdot \dot{q} + \dot{\theta}) \ .$$

Since L_k is quadratic and positive definite in velocities, the Euler-Lagrange equations are geodesic equations in Q_k for the metric for which L_k is the kinetic energy. Moreover p is a conserved quantity so that we can define $p = e/c = const$ which leads to an Hamiltonian that differs from the previous one by a term $(1/2)p^2$.

This construction generalizes to the case of a Yang-Mills field where ω becomes the connection and its curvature measures the field strength (that reproduces $B = \nabla \times \mathrm{A}$).

Kaluza Klein is one view point on the motion of a particle travelling through a Riemmanian manifold while under the influence of a Gauge field $\mathbf{A}$(such a particle is also called the Yang-Mills particle). There is an alternative view point which leads to a generalization of the Kaluza-Klein construction and ends in the so called Wong's equations. These equations look particularly interesting because (as we will see later) they have the same form as the equations resulting from an Optimal Control problem.

Be Q the configuration space, let G be a Lie group that acts on Q and $S = Q/G$ be the quotient space. Let $\pi : Q \to S$ be the map that assigns to each configuration q $x = \pi(q)$. S is a smooth manifold and π gives Q the structure of a principal bundle. The Wong's equations are equations for a curve $e(t)$ in the coadjoint bundle $\mathrm{g}^*(Q)$ which is a vector bundle over S with fiber g^* the dual of the Lie algebra of the group G. This bundle is an associated bundle to Q and the points $e \in \mathrm{g}^*(Q)$ are called *charges*. Be $x(t) = \pi(e(t)) \in S$ and $\dot{x} \in TS$; be D the connection on $\mathrm{g}^*(Q)$ induced by the connection A on Q ($De = de + ad^*(A)e$). Let ∇ be the Riemannian Levi-Civita connection on S induced by the metric k. Let $F = dA + [A, A]$ denote the curvature of A viewed as a two form on S with values in the adjoint bundle. Then $e \cdot F(\dot{x}, \cdot)$ is a one form along x (a force) and $e \cdot F(\dot{x}, \cdot)^\sharp$ is a vector field along x (and $\sharp$ is the operation of rising indeces with respect to the metric k on S). Then the Wong's equations are

$$\nabla_{\dot{x}}\dot{x} = e \cdot F(\dot{x}, \cdot)^\sharp \tag{1}$$

$$\frac{De}{dt} = 0 \tag{2}$$

If e were 0 the first equation would say that x is a geodesic on S.

Roughly speaking the elements in the coadjoint bundle are the lagrange multipliers which enforce the nonholonomic constraints.In the case of a magnetic field these forces of constraint are due to the presence of a magnetic field which forces the charge to moove on a constrained path (a circle in the case of uniform field B).

Equations (2) can be rewritten in coordinate expression as

$$\dot{p}_\alpha = -\lambda_a F^a_{\alpha\beta}\dot{x}^\beta - \frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial r^\alpha} p_\beta p_\gamma \tag{3}$$

$$\dot{\lambda}_b = -\lambda_a C^a_{db} A^d_\alpha \dot{r}^\alpha \tag{4}$$

where $\lambda(t) \in \mathbf{g}^*(Q)$ ~~places~~ plays the role of e, $g_{\alpha\beta}$ is the local representation of the metric in the base space S:

$$\frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \frac{1}{2}\|\dot{r}\|^2$$

while $g^{\beta\gamma}$ is the inverse of the matrix $g_{\alpha\beta}$ and p_α is defined by:

$$p_{\alpha = \frac{\partial l}{\partial \dot{x}^\alpha} = g_{\alpha\beta}\dot{x}^\beta}$$

where l is the reduced Lagrangian, A^d_α are the componets of A, $F^a_{\alpha\beta}$ are the components of F.

The Isoholonomic Problem as an Optimal Control Problem

Given a deformable body, what is the most efficient way to deform itself so as to achieve the desired reorientation? This problem is usually refered to as the Cat's Problem since it is the problem faced by the upside down zero angular momentum cat in free fall.

In this setting the spaces Q and S defined geometrically above become S the shape space and Q is the configuration space in which two objects can have the same shape but be rotated by mean of $g \in G$; in general G will be the group of rigid motions. The map π assigns to each configuration the respective shape.

Def. A vector v in T_qQ which is tangent to the group orbit $G \cdot q$ through $q \in Q$ is said to be vertical at q. Vertical vectors represent infinitesimal rigid rotations of our deformable body. A vector v in T_qQ is *horizontal* (HOR) if it is orthogonal to the group orbit through q. In the problem of the cat it is easily shown that $v \in HOR$ iff its angular momentum $M(q, v) = 0$.

Since a cat starts and ends its fall approximatively with the same shape, then we can say that it describes a loop in the shape space.Then we can define the

ISOHOLONOMIC PROBLEM: find the shortes loop in the shape space with a given holonomy. In this case the holonomy is the constraint that $\dot{q} \in HOR$.

This problem can be restated as a control problem for the reorientation of a deformable body. Think to a cat or a gymnast in free fall with a zero angular momentum: the problem is to reorient itself by changing its shape. The objective here is to control the net reorientation of the body, the control variables are the deformations dx of the shape x. Shape deformations are in

turn implemented by torques or linear forces applied to the joints.In this way we can view the tangent vectors $\dot{x}$ to the shape space as control variables $\dot{x} = u$. Therefore the problem is to find the minimum L_2-norm control law $u(t)$ that leads the system from configuration q_0 for $t = 0$ to configuration q_1 for $t = 1$ while mantaining feasibility of the solution ($A\dot{q} = 0$, i.e.$\dot{q} \in HOR$).We can rewrite the problem in formulae as:
fixed 2 points $q_1, q_2 \in Q$, among all curves $q(t) \in Q$, $0 \leq t \leq 1$ such that $q(0) = q_0$and $q(1) = q_1$ and $\dot{q}(t) \in HOR$, find the curve such that the energy of the base space curve

$$\frac{1}{2}\int_0^1 \|\dot{x}\|^2 dt$$

is minimized. Let's show how the solution of this minimization problem gives again the Wong's equations establishing a nice relation ~~analogy~~ between this problem and a charge in a Yang-Mills field.

proof. By general principles of variational calculus, given an optimal solution $q(t)$, there exist a lagrange multiplier $\lambda(t)$, such that the new function

$$\frac{1}{2}\int_0^1 (\|\dot{x}\|^2 + 2 < \lambda(t), A\dot{q}(t) >)dt$$

has a critical point at this curve. Considering the integrand as a Lagrangian and applying the reduced Euler-Lagrange equations to the reduced Lagrangian

$$l(x, \dot{x}, \Omega) = \frac{1}{2}\|\dot{x}\|^2 + < \lambda, \Omega >$$

where $\Omega = A\dot{q}$, we find

$$\frac{\partial l}{\partial \dot{x}^\alpha} = g_{\alpha\beta}\dot{r}^\beta \qquad \frac{\partial l}{\partial x^\alpha} = \frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial x^\alpha}\dot{x}^\beta\dot{x}^\gamma$$

$$\frac{\partial l}{\partial \Omega_a} = \lambda_a$$

and since the constraint is $\Omega = 0$ the reduced Euler Lagrange equations become

need to recall what these are.

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = -\lambda F^a_{\alpha\beta}\dot{x}^\beta$$

$$\frac{d}{dt}\lambda_b = -\lambda_a E^a_{\alpha b}\dot{x}^\alpha = -\lambda_a C^a_{db} A^d_\alpha \dot{r}^\alpha$$

since we have

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = \dot{p}^\alpha - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\dot{x}^\beta \dot{x}^\gamma =$$

$$= \dot{p}^\alpha + \frac{1}{2}\frac{\partial g^{k\sigma}}{\partial x^\alpha} g_{k\beta} g^{\sigma\gamma}\dot{x}^\beta \dot{x}^\gamma =$$

$$= \dot{p}^\alpha + \frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial x^\alpha} p_\beta p_\gamma$$

so that we obtain the Wong's equations (4).

In order to better see the analogy between the isoholonomic problem and the problem of a charge in a magnetic field, let's consider an example where $\dot{\lambda} = 0$ so that it is a conserved quantity which places the role of the charge e of a particle mooving in a classical magnetic field (note that both elements belong to the cotangent bundle).

Brockett example

This example is a proto~~tipe~~ type for the falling cat problem and a variety of optimal steering problems. Consider the control system in $\mathbb{R}^3$ (Heisemberg system):

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{y} &= u_2 \\ \dot{z} &= x u_2 - y u_1 \end{aligned} \tag{5}$$

where u_1,u_2 are control inputs. By controllability theorem it is known that it is possible to choose the controls in order to steer trajectories between any problem that is to find the horizontal curve joining p to q whose length is $d(p,q)$.Therefore the zero angular momentum cat problem is precisely the problem of finding minimizimg sub-Riemann geodesics.

Conclusions

In this report I have given a qualitative overview of how the same problem (isoholonomic problem) can be reformulated in different ways so to lead to interesting relations ~~analogies~~ with other dynamical systems(a paricle in a Yang-Mills field) which are phisically completely different. Those analogies have lead to a reinterpretation of the lagrange multipliers of the Optimal Control problem as charges of particles mooving in a Yang-Mills field, and to a reinterpretation (on the other oder side) of the magnetic forces as forces of constraint acting on a "potential free" system whose only energy is the kinetic energy. Finally it has been briefly described how the Optimal Control formulation is equivalent to the problem of finding geodesics in sub Riemann geometry.

References

[1] R.Montgomery Gauge Theory of The Falling Cat,*Fields Ins. Commun.* **1**, 1993, pp. 193-218

[2] R.Montgomery Canonical Formulation of a Classical Particle in a Yang-Mills Field and Wong's Equations *Lett. Math. Phy.* **8**,1984, pp.59-67

[3] R.Montgomery Optimal Control of Deformable Bodies and Its Relation to Gauge Theory, *The Geometry of Hamiltonian Systems*,(T. Ratiu, ed.),Springer-Verlag. pp. 404-438 [4] R.Montgomery Isoholonomic Problems and Some Applications *Commun. Math. Phys.* **128**, 1990, pp. 565-592

[5] J. E. Marsden Optimal Control and Stabilization of Balance Systems *Park*

City Lectures on Mechanics, Dynamics, and Symmetry, 1997, pp.43-52

[6] J.E. Marsden, and T. S. Ratiu *Introduction to Mechanics and Symmetry* Springer 1999

[7] W.Klingenberg *A Course in Differential Geometry* Springer-Verlag 1978