The Search for Stable Equilibria on Coadjoint Orbits and Applications to Dissipative Processes

MA 275
Spring 1994
Project Report

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May 24, 1994

Very nice report. Thorough & very well written. A.
JM.

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1 The General Coadjoint Orbit Stable Equilibrium Search Program

The general ideas surrounding the equilibria search algorithms discussed in this report are as follows:

1. Consider only the isolated equilibria of Hamiltonian systems which are global extrema of the Hamiltonian (energy)
2. From symplectic geometry we know that for Hamiltonian systems, a state starting on a coadjoint orbit, will remain in that orbit for all time. We also know that the coadjoint orbits are the level sets of Casimirs.
3. We know further that for Hamiltonian systems, the state evolves on the intersection of a coadjoint orbit and a level set of the energy of the system.
4. Let us assume that (at least one) isolated extrema exist on the coadjoint orbit of choice.
5. Modify the equations of motion so that
   (a) The system state does not leave its initial coadjoint orbit
   (b) The system equilibria are preserved
   (c) The energy of the system monotonically increases or decreases.

   This implies that the Casimirs must be preserved by the modification of the Hamiltonian dynamics ($\dot{C} = 0$).
6. Following this program, the modified system will then evolve to an isolated extrema of energy on a chosen coadjoint orbit (level set of a Casimir).
7. The the procedure preserves coadjoint orbits, but in general it will change the topology of the level sets of energy on the coadjoint orbit. (See Figure 1)

There are at least two good physically based examples for following this general coadjoint orbit stable equilibrium search program:

1. **Fluid Dynamics**

   In the case of fluid dynamics, it is desirable to identify equilibrium flow fields of a given vortex topology, through the use of computer simulations. The equilibrium search must then occur on a specified coadjoint orbit. (Kelvin 1887) (Vallis, Carnevale & Young 1989) (Vallis, Carnevale & Shepherd 1989) (Shepherd 1990) (Carnevale & Vallis 1990)

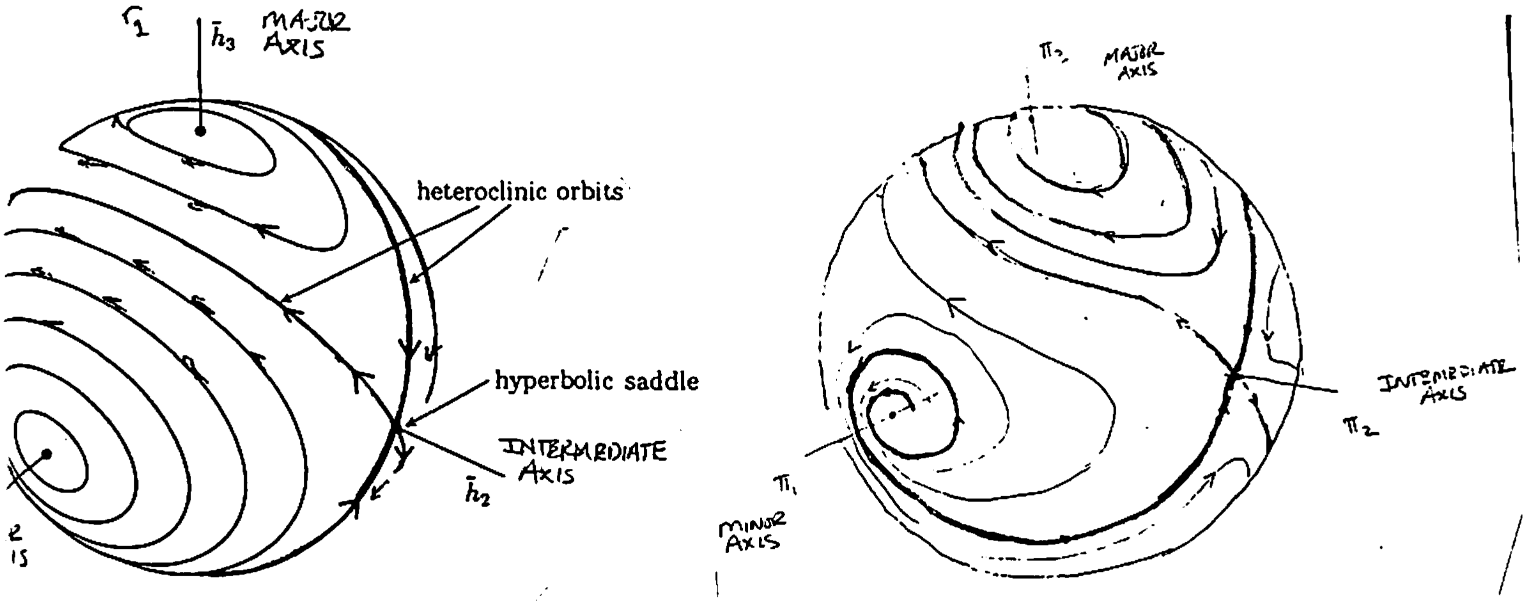

Figure 1: Hamiltonian and Energy-Casimir Modified Dynamics for the Rigid Body

2. **Rigid Body Dynamics**

In the course of modeling quasi-rigidity (energy loss) for a freely rotation body in space, it is necessary that any initial angular momentum of the body be preserved during the energy loss process. It is then necessary to find a energy loss mechanism which preserves the initial coadjoint orbit: The energy loss must proceed on the initial angular momentum sphere. (Rimrott 1989) (Kammer & Gray 1993) (Vallis, Carnevale & Shepherd 1989) (Bloch, Krishnaprasad, Marsden & Ratiu 1991) (Coleman 1993)

We document the development of the research this area. The equilibrium search method outline above, was motivated by the stability analysis program started by Arnold (Arnold 1965*a*) (Arnold 1965*b*) (Arnold 1966). It appears that the stable equilibrium search program was originally developed by fluid dynamicists in an effort to search for stable two dimensional flows (Vallis, Carnevale & Young 1989) (Carnevale & Vallis 1989) (Shepherd 1990) (Vallis, Carnevale & Shepherd 1989) (Carnevale & Vallis 1990). It has since inspired applications in finite dimensions to model dissipation of energy (Vallis, Carnevale & Shepherd 1989) (Bloch et al. 1991) (Coleman 1993).

It is of historical interest to note that Kelvin knew, in the context of Hamiltonian fluids, that stationary states are points for which the energy is stationary stationary with respect to variations on a given coadjoint orbit (Vallis, Carnevale & Young 1989, p 134) (Kelvin 1887). Specifically Kelvin knew that the first variation of the functional $H + C$ must vanish at a stationary point (Vallis, Carnevale & Young 1989, p 136).

Lagrange and Dirichlet knew that in order to analyze the stability of a Hamiltonian, it is necessary to look at the second variation $\delta^2 H$ of H (Marsden & Ratiu 1993, p 30). Arnold, in some extending the stability work of Liapunov with the introduction of key convexity estimates, and in another sense extending the analytical program of Lagrange–Dirichlet, was able to establish the concept of formal stability for two dimensional fluids, by analyzing $\delta^2(H+C)$.

It is interesting to note that more than 100 years after Kelvin's observation, with the motivation provided by work done Arnold over almost 30 years ago, contemporary researchers are actively using computer computations to search for stationary flows.

2 Vallis, Carnevale & Young (1989)

Motivation

In finite dimensions, fairly straight forward algebraic calculations will yield the equilibrium points of Hamiltonian systems. Once the equilibrium (stationary) points are known, one can assess the stability of the equilibria by proceeding with the energy-Casimir-momentum stability analysis proposed by Marsden & Ratiu (1993, pp 31–33), or with the energy-Casimir stability analysis given in Holm, Marsden, Ratiu & Weinstein (1985, pp 7–11).

In infinite dimensions, such as Eulerian fluids, it is not easy to find *a priori* equilibrium flows. With this fact in mind Vallis, Carnevale & Young (1989) were motivated to find a numerical algorithm that would search for stationary points [1] of perfect fluid flows.

Vallis, Carnevale & Young (1989) are in some sense aware of the fact that Eulerian fluid flows evolve on the intersection coadjoint orbits (isovortical surfaces which preserve the vortex topology), and level surfaces of constant energy. (See Figure 2)

With this knowledge, Vallis, Carnevale & Young (1989) embark on a program to find a way of modifying the Eulerian equations of fluid flow such that

1. The modified dynamics are restricted to the initial coadjoint orbit.
2. Stationary points are preserved.
3. The energy of the system monotonically increases or decreases.

The modified dynamics should then evolve to stable stationary points using Arnold stability arguments (Arnold 1965*a*) (Arnold 1965*b*) (Arnold 1966). There are many technical details regarding equilibria, which have been left out in this short review of Vallis, Carnevale & Young (1989), but they shall be addressed in Chapter 6.

[1] In the fluid mechanics literature equilibrium flow fields are sometimes called *stationary states* or *stationary points* and we will use this terminology interchangeably when referring to the equilibria of Eulerian fluid flows in this report.

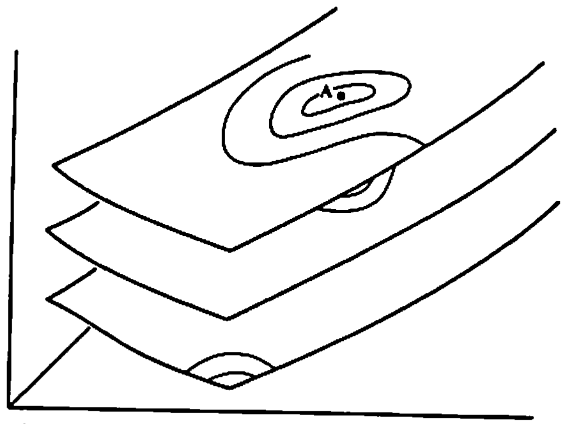

FIGURE 1. Schema of sheets in the fluid phase-space (after Arnol'd 1965*b*). Each sheet is isovortical, meaning that evolution on it conserves circulation. The lines are representations of energy surfaces, of lower dimension, embedded in the isovortical surfaces. Since the evolution of an Euler fluid is confined to a particular energy surface on a given sheet, extremum points (e.g. point A) are stable.

Figure 2: Isovortical Surfaces, Level Sets of Energy and Equilibria

Modifying the Dynamics

Vallis, Carnevale & Young (1989) require modified dynamics in which the energy in monotonically dissipated or generated. The simple use of viscosity is inappropriate, since the evolving flow will not remain on its initial coadjoint orbit (isovortical sheet). Modified dynamics can be done if the system is evolved by an advective process, since the original Hamiltonian system evolves by an advective process which preserves topological invariants such as vorticity. Modification of the original Hamiltonian advective velocity field $\boldsymbol{u}$ is suggested as the only appropriate modification of the vorticity equation. Vallis, Carnevale & Young (1989, p 137) propose and prove by shear calculation, that the modified vector field $\tilde{\boldsymbol{u}}$, given by

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \qquad (1)$$

preserves topological invariants and monotonically dissipates or generates energy.

Carnevale & Vallis (1990) call the flow field $\tilde{\boldsymbol{u}}$ *pseudo-advective*. The new advecting flow $\tilde{\boldsymbol{u}}$ field is the sum of the original Hamiltonian flow field and a scalar factor of α times the time rate of change of the original Hamiltonian flow field.

This seems to be the first appearance of modified dynamics, for Hamiltonian systems. And Vallis, Carnevale & Young (1989) conjecture that their method is rather general as evidenced by the following quote:

> The method appears sufficiently general that application in other fields seems likely. Consider, for example, any dynamical system
>
> $$\dot{\boldsymbol{x}} = \boldsymbol{f}(\{x_i\}) \qquad (7.1)$$

where $\boldsymbol{x}$ is a multi-dimensional state vector, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Suppose the system has a conserved quantity $\sum x_i^2$, which we call energy, obtained by taking the dot product of (7.1) with $\boldsymbol{x}$. Then to form a modified set of dynamics for the system (7.1) we can replace the ith component of $\boldsymbol{x}$, namely x_i, by $x_i + \alpha\dot{x}_i$ in all the right-hand sides of the equations. Then, in general, the energy of the system will change monotonically by $\alpha\dot{x}_i^2$. The trick, of course, is to choose the form of the equations and the variable in such a way that other invariants remain preserved, so that the solution may evolve to a stable steady state. (Vallis, Carnevale & Young 1989, pp 151–152).

No proof is this conjecture is offered in Vallis, Carnevale & Young (1989). Ultimately, Shepherd (1990) will show that the general form of modified dynamics suggested by Vallis, Carnevale & Young (1989), for *any* Hamiltonian system, is of the form:

$$\begin{aligned} u_t &= X_H(u) + J\alpha X_H(u) \\ u_t &= J\frac{\partial\mathcal{H}}{\partial u} + J\alpha J\frac{\partial\mathcal{H}}{\partial u} \end{aligned} \tag{2}$$

where J is a skew-symmetric transformation from function space $\{u\}$ to $\{u\}$, satisfying the Jacobi and Leibniz identities; $X_H = J\frac{\partial\mathcal{H}}{\partial u}$ are the Hamiltonian equations of motion of the system; α is a symmetric transformation with $(u, \alpha u)$ of definite sign for all u and the appropriate inner product $(\cdot,\cdot)$.

Not having the formal Hamiltonian structure developed by Shepherd (1990), Vallis, Carnevale & Young (1989) have to show "by hand" that their proposed pseudo-advective flow accomplishes the objectives of their equilibrium search program.

Example: Modifying the Advecting Velocity Field

To illustrate the application of the pseudo-advection modified dynamics proposed by Vallis, Carnevale & Young (1989), the method is applied to the dynamics of incompressible flow (Vallis, Carnevale & Young 1989, pp 137–138).

Assume an incompressible flow of constant (unity) density. The Euler equations of motion of the fluid are

$$\frac{\partial\boldsymbol{u}}{\partial t} - \boldsymbol{u}\times\boldsymbol{\omega} = -\nabla b \tag{3}$$

with

$$\nabla\cdot\boldsymbol{u} = 0 \tag{4}$$

$$\begin{aligned}\boldsymbol{\omega} &= \nabla \times \boldsymbol{u} \\ b &= p + \frac{1}{2}u^2\end{aligned}$$

Assume that 3 and 5 are valid in a domain $\mathcal{D}$ which may be finite or infinite, but in which there is, in any case, no contribution to any of the integrals in the following manipulations from the boundaries or from infinity.

Energy conservation follows by taking the dot product with $\boldsymbol{u}$ and integrating over $\mathcal{D}$. The nonlinear term vanishes, and the term $\boldsymbol{u} \cdot \nabla b = \nabla \cdot (\boldsymbol{u}b)$, similarly disappears provided there are no boundary contributions. Therefore

$$\begin{aligned}E &= \frac{1}{2}\int_{\mathcal{D}} u^2 dV \\ \frac{dE}{dt} &= 0\end{aligned} \tag{5}$$

Consider the following equations describing the evolution of the pseudo-advective modified dynamics:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \tilde{\boldsymbol{u}} \times \boldsymbol{\omega} = -\nabla \tilde{b} \tag{6}$$

where

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \tag{7}$$

Upon substituting 6 into 7 one finds

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \nabla \times \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) \tag{8}$$

where $\tilde{b} = p + \frac{1}{2}\tilde{u}^2$

Close the set of equations by adding

$$\begin{aligned}\boldsymbol{\omega} &= \nabla \times \boldsymbol{u} \\ \nabla \cdot \tilde{\boldsymbol{u}} &= \nabla \cdot \boldsymbol{u} = 0\end{aligned} \tag{9}$$

The energy budget of the closed set 6, 7, 8 and 10 is obtained by taking the dot product of 6 with $\tilde{\boldsymbol{u}}$ for 7, or $\boldsymbol{u}$ for 8, and integrating over $\mathcal{D}$. The nonlinear terms vanish and we have

$$
\begin{aligned}
E &= \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{u}^2 dV \\
\frac{dE}{dt} &= -\alpha \int_{\mathcal{D}} \boldsymbol{u}_t dV
\end{aligned}
\tag{10}
$$

or

$$
\frac{dE}{dt} = -\alpha \int_{\mathcal{D}} [\nabla(\boldsymbol{u} \times \boldsymbol{\omega})]^2 dV \tag{11}
$$

Since the integral in 11 is definite, the energy in the $\boldsymbol{u}$-field monotonically decreases, or increases, depending on the sign of α. The energy will change monotonically until a steady state is reached. Whenever there is unsteady motion, energy changes monotonically. If and when a steady state is reached, the modified dynamics become identical to the original Hamiltonian dynamics as the extra terms of the pseudo-advective field vanish.

Steady solutions (equilibrium states) of 6 satisfy

$$
\boldsymbol{u} \times \boldsymbol{\omega} = -\nabla b
$$

with the constraints $\boldsymbol{\omega} = \nabla \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u} = 0$.

Because of the monotonic change in energy, whenever there is motion, the fluid must either tend toward a state of rest or infinite energy, or to a non-trivial solution of the Euler equations. This motion is achieved with conserved circulation, helicity and potential vorticity as shown in Vallis, Carnevale & Young (1989, §3,pp 138–141). Therefore, the mapping from initial to final state is isovortical: The mapping stays on the initial coadjoint orbit as is required by the general coadjoint orbit equilibrium search program.

Other Examples

Vallis, Carnevale & Young (1989) literally repeat the above illustrated process, of substituting a pseudo-advective flow field

$$
\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \frac{\partial \boldsymbol{u}}{\partial t}
$$

into the equations of motion and analyzing the energy budget, for the following flows:

1. Two-Dimensional Flows (Vallis, Carnevale & Young 1989, pp 141-142)
2. Quasi-Geostrophic Flows (Vallis, Carnevale & Young 1989, pp 142-143)
3. Stratified Flow and the Shallow-Water Equations (Vallis, Carnevale & Young 1989, pp 143-144)

A numerical example of a two-dimensional, irregular shaped patch of constant vorticity is shown to evolve to its stable maximum energy configuration, a circle. Of course, the original shape of the vortex patch was topologically equivalent to a circular patch, since the connectivity, boundary and area are topological invariants which must remain unchanged in the flow in finite time (See Figure 3).

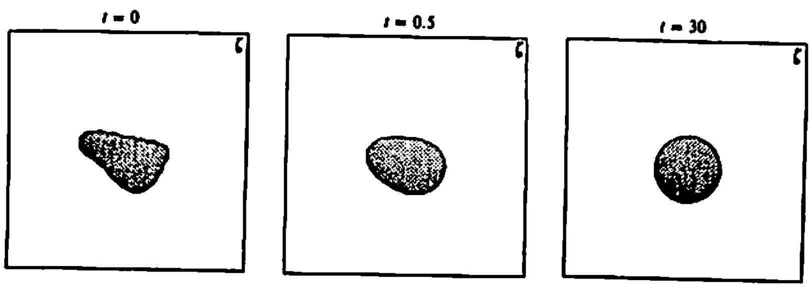

Figure 3: Pseudo-Advective Relaxation to Equilibrium

Moffatt's Approach

The equilibrium search program given in Moffatt (1985) is similar in philosophy to that given in Vallis, Carnevale & Young (1989). However the Vallis, Carnevale & Young (1989) procedure preserves vortex topology (coadjoint orbits), whereas the Moffatt (1985) procedure preserves streamline topology (the topology of the energy orbits on the coadjoint orbit). The Moffatt (1985) procedure both finds stable and unstable solutions. The Vallis, Carnevale & Young (1989) program will generally ignore saddle points. Moffatt's method is useful because unstable solutions are sometimes of interest (Vallis, Carnevale & Young 1989, p 136). Moffatt's approach is not reviewed in this document, but is it of future research interest to the author of this report!

3 Shepherd (1990)

Shepherd (1990) generalizes and extends the work of Vallis, Carnevale & Young (1989), by stating and using a general energy-Casimir-momentum method. Shepherd's Hamiltonian formalism appears to be inspired in part by Holm et al. (1985).

Non-canonical Hamiltonian dynamical systems generally possess integral invariants known as Casimir functionals. In the case of the Euler equations for a perfect fluid, the Casimir functionals correspond to the vortex topology, whose invariance derives from the particle relabeling symmetry of the underlying Lagrangian equations of motion [2]

Vallis, Carnevale & Young (1989) present a specific algorithm for finding steady states of the Euler equations that represent extrema of energy subject to a given vortex topology (coadjoint orbit/isovortical surface). Shepherd (1990) presents a general method for modifying *any* Hamiltonian dynamical systems which will systematically increase or decrease the energy of the system while preserving all of the Casimir invariants. Incorporating momentum into the extremization procedure allows the algorithm to find find steadily-translating as well as steady stable states.

Hamiltonian Dynamical Systems

Consider a general continuous Hamiltonian dynamical system, whose governing equations are written in symplectic form as

$$u_t = J\frac{\delta \mathcal{H}}{\delta u} \tag{12}$$

The dependent variable u is a function of time t and of position $\boldsymbol{x}$ with some domain $\mathcal{D}$. In finite dimensions u is a function of t only. In general, u_t is the partial derivative of u with respect to t; $\mathcal{H}(u)$ is the Hamiltonian functional, which is usually the total energy of the system. $\delta\mathcal{H}/\delta u$ is the (Fréchet) functional or variational derivative of $\mathcal{H}$, defined as usual by

$$\delta\mathcal{H} \equiv \mathcal{H}(u+\delta u) - \mathcal{H}(u) = \left(\frac{\delta\mathcal{H}}{\delta u}, \delta u\right) + O(\delta u^2) \tag{13}$$

for admissible and arbitrary variations δu which vanish at the domain boundary. $(\cdot,\cdot)$ is the relevant inner product for the functions space $\{u\}$. J is a skew-symmetric transformation from $\{u\}$ to $\{u\}$, satisfying

$$(u, Jv) = -(Ju, v) \tag{14}$$

as well as the Jacobi and Leibniz identities.

Integral Invariants of Hamiltonian Systems

The Hamiltonian dynamical system described by 12 generally possesses the following three integral invariants:

1. The Hamiltonian $\mathcal{H}$
2. Momentum Invariants
3. Casimir Invariants associated with the kernel of the operator J

[2] See Marsden & Ratiu (1993, §1.4, p 13) for a discussion on particle relabeling symmetry.

The Hamiltonian Invariant

The Hamiltonian functional is an integral invariant. This is shown by calculating the time derivative of $\mathcal{H}$:

$$\frac{d\mathcal{H}}{dt} = \left(\frac{\delta\mathcal{H}}{\delta u}, u_t\right) = \left(\frac{\delta\mathcal{H}}{\delta u}, \boldsymbol{J}\frac{\delta\mathcal{H}}{\delta u}\right) = 0 \tag{15}$$

which is true by by the skew–symmetry of $\boldsymbol{J}$.

Momentum Invariants

Momentum invariants are related by Noether's theorem to the spatial (translational) symmetries of the Hamiltonian. If $\mathcal{H}$ is invariant under translations in x then the associated momentum functional $\mathcal{M}$ is defined (to within a Casimir) by

$$-u_x = \boldsymbol{J}\frac{\delta\mathcal{M}}{\delta u} \tag{16}$$

The momentum functional $\mathcal{M}$ is conserved by the dynamics of the Hamiltonian system described by 12. This is shown by calculating the time derivative of $\mathcal{M}$:

$$\frac{d\mathcal{M}}{dt} = \left(\frac{\delta\mathcal{M}}{\delta u}, u_t\right) = \left(\frac{\delta\mathcal{M}}{\delta u}, \boldsymbol{J}\frac{\delta\mathcal{H}}{\delta u}\right) = -\left(\boldsymbol{J}\frac{\delta\mathcal{M}}{\delta u}, \frac{\delta\mathcal{H}}{\delta u}\right) = \left(u_x, \frac{\delta\mathcal{H}}{\delta u}\right) = 0 \tag{17}$$

Casimir Invariants

Casimir functionals $\mathcal{C}(u)$ are associated with the kernel of the operator $\boldsymbol{J}$. Casimirs are solutions of the equation

$$\boldsymbol{J}\frac{\delta\mathcal{C}}{\delta u} = 0 \tag{18}$$

Casimirs are conserved by the dynamics of the Hamiltonian system 12.

$$\frac{d\mathcal{C}}{dt} = \left(\frac{\delta\mathcal{C}}{\delta u}, u_t\right) = \left(\frac{\delta\mathcal{C}}{\delta u}, \boldsymbol{J}\frac{\delta\mathcal{H}}{\delta u}\right) = -\left(\boldsymbol{J}\frac{\delta\mathcal{C}}{\delta u}, \frac{\delta\mathcal{H}}{\delta u}\right) = \left(0, \frac{\delta\mathcal{H}}{\delta u}\right) = 0 \tag{19}$$

For fluid systems the Casimir invariants include 'topological invariants' such as helicity as well as entropy, vorticity and potential vorticity. Casimirs can also include the mass of the system, where appropriate (Moffatt 1989) (Zagdeev, Tur & Yanovsky 1989).

The existence of non-trivial Casimirs depends on the non-canonical nature of the Poisson bracket for the Hamiltonian system (Littlejohn 1982). The Eulerian representation of fluids with its particle relabeling symmetry associated with the passage from material (Lagrangian) to spatial (Eulerian) coordinates, is in general non-canonical, and therefore will generally possess non-trivial Casimirs.

There are exceptional Hamiltonian fluids systems which possess no Casimir invariants. Irrotational water wave can be written in canonical Hamiltonian form, and therefore the system possesses no Casimir invariants (Shepherd 1990, p 585).

Poisson Brackets and Hamiltonian Systems

The Hamiltonian system 12 can equivalently be represented by the following Poisson bracket system:

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \tag{20}$$

where $\mathcal{F}$ is any functional of u whose functional derivative is well-defined, and $\{\cdot,\cdot\}$ is the generally non-canonical Poisson bracket defined by

$$\{\mathcal{F}, \mathcal{G}\} \equiv \left(\frac{\delta \mathcal{F}}{\delta u}, J\frac{\delta \mathcal{G}}{\delta u}\right) \tag{21}$$

With this definition the Poisson bracket inherits the needed skew, Jacobi and Leibniz properties.

In the Poisson structure formulation of Hamiltonian systems, a Casimir $\mathcal{C}$ is any functional which satisfies

$$\{\mathcal{C}, \mathcal{G}\} = 0 \tag{22}$$

for all admissible functionals $\mathcal{G}$. It is easy to see that in this formulation, Casimirs are associated with the degeneracy of the Poisson bracket, but from 21 we see that this is equivalent to asserting that Casimirs are intimately associated with the kernel of the operator J.

Modified Dynamics: Energy-Casimir Method

Consider the modification of the Hamiltonian system 12

$$u_t = J\frac{\delta \mathcal{H}}{\delta u} + J\alpha J\frac{\delta \mathcal{H}}{\delta u} \tag{23}$$

where α is a symmetric transformation with $(u, \alpha u)$ of definite sign for all u.

Notes on the Form of α

The factor α may be taken to be a constant, single-signed diagonal matrix who entries will generally have differing dimensions. More general forms are possible, such as the iterated Laplacian forms given in Vallis, Carnevale & Young (1989, pp 142–143).

Notes on $X_H = J\frac{\delta \mathcal{H}}{\delta u}$

The first term of 23

$$J\frac{\delta \mathcal{H}}{\delta u}$$

which represents the original Hamiltonian vector field, is not needed to ensure preservation of the coadjoint orbit and monotonic increase or decrease of energy. This will be illustrated through the energy budget and Casimir preservation calculations to be performed shortly. Nonetheless, for clarity and generality, it is maintained through the rest of this analysis.

Casimirs are Preserved Under the Modified Dynamics

The Casimir invariants of the original Hamiltonian system 12 are also Casimir invariants of the modified system 23.

$$\begin{aligned} \frac{dC}{dt} &= \left(\frac{\delta C}{\delta u}, u_t\right) = \left(\frac{\delta C}{\delta u}, J\frac{\delta \mathcal{H}}{\delta u} + J\alpha J\frac{\delta \mathcal{H}}{\delta u}\right) \\ &= -\left(J\frac{\delta C}{\delta u}, \frac{\delta \mathcal{H}}{\delta u} + \alpha J\frac{\delta \mathcal{H}}{\delta u}\right) = \left(0, \frac{\delta \mathcal{H}}{\delta u} + \alpha J\frac{\delta \mathcal{H}}{\delta u}\right) \\ &= 0 \end{aligned} \tag{24}$$

The Hamiltonian Monotonically Increases or Decreases and Steady Solutions are Preserved

Under the modified dynamics 23 the Hamiltonian is no longer an invariant.

$$\frac{d\mathcal{H}}{dt} = \left(\frac{\delta \mathcal{H}}{\delta u}, u_t\right) = \left(\frac{\delta \mathcal{H}}{\delta u}, J\frac{\delta \mathcal{H}}{\delta u} + J\alpha J\frac{\delta \mathcal{H}}{\delta u}\right) = -\left(J\frac{\delta \mathcal{H}}{\delta u}, \alpha J\frac{\delta \mathcal{H}}{\delta u}\right) \tag{25}$$

The right hand side of 25 is of definite sign, and is non-zero unless

$$J\frac{\delta \mathcal{H}}{\delta u} = 0 \tag{26}$$

We conclude from 25 and 26, that the energy of the modified system increases or decreases monotonically depending on the sign of alpha, and that the steady solutions (equilibria) of the modified systems are those of the original Hamiltonian system.

Modified Dynamics: Energy-Casimir-Momentum Method

If the Hamiltonian dynamical system 12 is invariant under translations in x, the one may expect the existence of steadily-translating solutions of the form $u = U(x - ct)$ The other spatial variables are implicit, and c is the translation velocity in the x-direction. These solutions satisfy

$$U_t + cU_x = 0 \tag{27}$$

Allowing J to operate on 27 we derive the expression

$$\left(\frac{\delta \mathcal{H}}{\delta u} - c\frac{\delta \mathcal{M}}{\delta u}\right)|_{u=U} = 0 \tag{28}$$

The energy-Casimir modified dynamics algorithm 23 is generalized to the energy-Casimir-momentum method, which is capable of finding steadily-translating states, by the following modified dynamics:

$$u_t = J\frac{\delta}{\delta u}(\mathcal{H} - c\mathcal{M}) + J\alpha J\frac{\delta}{\delta u}(\mathcal{H} - c\mathcal{M}) \tag{29}$$

with c specified. Under evolution of the energy-Casimir-momentum modified dynamics 29, all the Casimirs of the original Hamiltonian system 12 are left invariant. The functional $\mathcal{H}-c\mathcal{M}$ will monotonically increase or decrease under the energy-Casimir-momentum modified dynamics, as illustrated by the calculation below:

$$\frac{d}{dt}(\mathcal{H} - c\mathcal{M}) = -\left(J\frac{\delta}{\delta u}(\mathcal{H} - c\mathcal{M}), \alpha J\frac{\delta}{\delta u}(\mathcal{H} - c\mathcal{M})\right) \tag{30}$$

The evolution process under the energy-Casimir-momentum modified dynamics will only stop if the system converges to a steadily translating state, satisfying 28. As with steady solutions, steadily translating solutions of 12 are necessarily conditional extrema of $\mathcal{H} - c\mathcal{M}$ for Casimir-preserving (coadjoint orbit preserving) variations. By construction, they may be true extrema. By treating c as a Lagrange multiplier, such states may also be regarded as extrema of $\mathcal{H}$ for fixed $\mathcal{M}$ and $\mathcal{C}$.

With the Hamiltonian structure and analysis given by Shepherd (1990), it is no longer necessary to conduct the explicit energy dissipation and Casimir preservation calculations performed in Vallis, Carnevale & Young (1989). One simply chooses either the energy-Casimir method or the energy-Casimir-momentum method, and calculates the appropriate modified dynamics using the (non-canonical) Poisson bracket of the system, and possibly a momentum functional.

Since the direct calculations in Vallis, Carnevale & Young (1989) can be tedious and are application specific. The formalism presented by Shepherd (1990) provides a clear, and perhaps elegant, setting for the analysis. The dependence of energy dissipation mechanism on the Hamiltonian/Poisson bracket structure of the equations of motion is made readily apparent.

Examples

Shepherd (1990) applies the energy-Casimir-momentum method to the following perfect-fluid systems:

1. Two-Dimensional Euler Flow
2. Three-Dimensional Euler Flow
3. Baroclinic Quasi-Geostrophic Flow Over Topography
4. Two-Dimensional Stratified Boussinesq Flow
5. Rotating Homogeneous Shallow-Water Flow
6. Three-Dimensional Rotating, Stratified, Compressible Flow of an Ideal Gas (the Meteorological Primitive Equations)

Although not explicitly stated, a comment by Shepherd (1990) suggests that the energy-Casimir-Momentum method could be applied to the finite dimensional example of a rigid body to find stable rotating states.

> It is obvious that the same approach, using a combination of energy and angular momentum, would similarly identify stable rotating states (details left to the reader). (Shepherd 1990, p 584)

4 Vallis, Carnevale & Shepherd (1989)

Vallis, Carnevale & Shepherd (1989) is an amalgamation of the work presented in Vallis, Carnevale & Young (1989) and Shepherd (1990). The general Hamiltonian formalism of the energy-Casimir method presented in Shepherd (1990) is restated; and the finite dimensional analogue of energy-Casimir method the appears explicitly (Vallis, Carnevale & Shepherd 1989, pp 430–432). The finite dimensional energy-Casimir modified dynamics appear below:

$$\dot{u} = \boldsymbol{J}\nabla H + \boldsymbol{J}\alpha\boldsymbol{J}\nabla H \tag{31}$$

where H is the Hamiltonian; $\boldsymbol{J}$ is a skew-symmetric operator (which may be a matrix) satisfying $(a, \boldsymbol{J}b) = -(\boldsymbol{J}a, b)$; and ∇ is the gradient operator.

Energy-Casimir modified dynamics are explicitly written for the dynamics of the rigid in body (Lagrangian) angular momentum coordinates. In notation consistent with Marsden & Ratiu (1993), the energy-Casimir modified dynamics for free rigid body appear as

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \alpha \boldsymbol{\Pi} \times (\boldsymbol{\Pi} \times \boldsymbol{\Omega}) \tag{32}$$

in Equation (3.6) of (Vallis, Carnevale & Shepherd 1989, p 434).

The incompressible flow example given in Vallis, Carnevale & Young (1989, pp 137–138) is presented once again. However, in this instance, instead of assuming a pseudo-advective flow field of the form

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \tag{33}$$

and proceeding with the analysis illustrated in Chapter 2, the Eulerian equations of motion are immediately placed into Hamiltonian form, and the modified dynamics are produced through the formal manner given in Shepherd (1990). Energy budgets and Casimir preservation follow immediately from the general analysis of modified dynamics presented in Shepherd (1990). (Vallis, Carnevale & Shepherd 1989, pp 435–437)

In Vallis, Carnevale & Shepherd (1989), it is suggested that energy-Casimir modified dynamics for the rigid body may be useful for modeling energy loss in real physical systems:

> It is interesting to speculate whether the above modified system has any physical significance other than being a means to evolve to a stable solution. If a spinning rigid body is released into superficially free motion, there are in practice likely to be dissipative mechanisms acting on the body, due for example to straining and flexing of its parts. Hence its kinetic energy may fall (the energy being converted to heat and radiated away), although its angular momentum is conserved, so that the body evolves to a state of minimum possible kinetic energy. (Vallis, Carnevale & Shepherd 1989, p 435)

One finds that in the field of spacecraft dynamics, energy loss in rotating satellite systems is modeled through the use of energy-sinks which dissipate kinetic energy while preserving angular momentum (Rimrott 1989) (Kammer & Gray 1993). This suggest that the energy-Casimir modified dynamics for the rigid body may have practical application.

The energy-Casimir modified dynamics for the free rigid body where known to Marsden through a communication from Krishnaprasad. [3] Marsden suggested to Coleman that these modified dynamics be compared to the energy-sink method given by Kammer & Gray (1993). Coleman verified that the energy-Casimir modified dynamics satisfy the energy-sink criteria

[3] Personal communication: Thursday, 12 May 1994

specified in Rimrott (1989), showing that the modified dynamics are a valid energy sink for modeling energy loss in rotating spacecraft systems (Coleman 1993). The original Hamiltonian and the energy-Casimir modified dynamics for the free rigid body are sketched in Figure 1.

Energy-sinks are factor added to the Hamiltonian equations of motion. The energy-sink given in Kammer & Gray (1993) and the term is not of the same form as that given by the energy-Casimir modified dynamics. This provides evidence that there may be a large class of modified dynamics which preserve coadjoint orbits while breaking constant energy symmetries and preserving equilibria.

In the next chapter it is shown that that energy-Casimir modified dynamics, as outlined in Shepherd (1990), belong to a class of double bracket energy dissipating systems.

5 Double Bracket Dissipative Systems

For finite dimensional systems, the energy-Casimir modified dynamics in Poisson bracket form, given by Vallis, Carnevale & Shepherd (1989, p 432) appear as

$$\dot{F} = \{F, H\} + (\nabla F, \boldsymbol{J}\alpha\boldsymbol{J}\nabla H) \tag{34}$$

where $(\cdot,\cdot)$ is the appropriate inner product, and $\{\cdot,\cdot\}$ is the Poisson bracket defined by $\{F,G\} = (\nabla F, \boldsymbol{J}\nabla G)$.

Vallis, Carnevale & Shepherd (1989) note the symmetric nature of the factor $(\nabla F, \boldsymbol{J}\alpha\boldsymbol{J}\nabla H)$ in the modified dynamics. There is no further discussion of this fact, other than stating

> The second term on the right hand side of [34] cannot generally be put in to a true Poisson bracket form because the operator $\boldsymbol{J}\alpha\boldsymbol{J}$ is symmetric whereas $\boldsymbol{J}$ is skew symmetric. Vallis, Carnevale & Shepherd (1989, p 432)

However, it turns out that the energy-Casimir modified dynamics of Shepherd (1990) and Vallis, Carnevale & Shepherd (1989) do fit into the framework of double bracket dissipative systems of the form

$$\dot{F} = \{F, H\}_{skew} + \{F, H\}_{symmetric} \tag{35}$$

In finite dimensional case 34, $\{\cdot,\cdot\}_{skew}$ is the (skew symmetric) Poisson bracket $\{\cdot,\cdot\}$ defined above; $\{F,H\}_{symmetric}$ is the symmetric Poisson bracket defined by $\{F,G\}_{symmetric} = (\nabla F, \boldsymbol{J}\alpha\boldsymbol{J}\nabla G)$. Double bracket dissipative dynamical systems are further discussed in Bloch, Krishnaprasad, Marsden & Ratiu (1994).

6 Technicalities of Equilibria

Within this report, we should point out some of the technicalities surrounding equilibria which are raised in Vallis, Carnevale & Young (1989) and Carnevale & Vallis (1990).

It is geometrically possible that a system perturbed from an equilibrium will not necessarily stay close to the equilibrium unless the stationary point is also an extremum of energy, and not a saddle point. (See Figure 4)

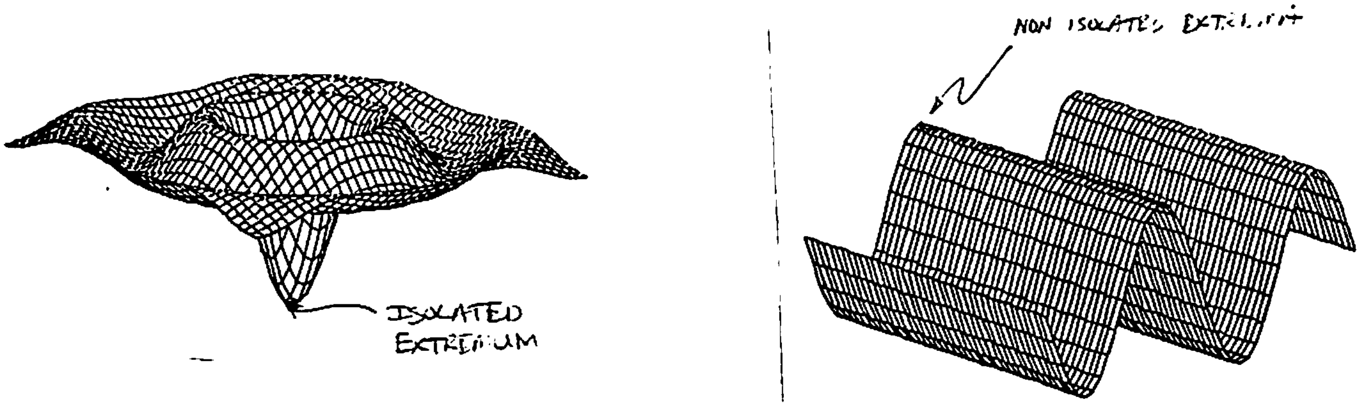

Figure 4: Non-Isolated and Isolated Extrema

However, if the stationary point is an extremum (and not a saddle), then the flow is stable in the sense of Liapunov. That is the size of the perturbation is bounded by the size of the initial perturbation for all time. This is the argument made rigorous for two-dimensional fluids by Arnold (1965*a*) Arnold (1965*b*). (Vallis, Carnevale & Young 1989, p 134)

There are cases when an energy extremum may not be stable. This occurs when an equilibrium is not isolated (See Figure 4). Then perturbations of the system around the maximum can cause it to move, along an energy contour, and the distance it can move may not be bounded. The second variation of the energy at these points is a singular quadratic form, since the variation of energy in the set of non-isolated equilibria is constant. (Vallis, Carnevale & Young 1989, p 134)

Unless stated, to avoid these pathologies, we restrict ourselves in this report to the analysis and discussion to equilibria which are non-singular, isolated, extrema of energy. In cases where isolated energy extrema exist, a stable steady flow or equilibrium can be found.

In two dimensions, Vallis, Carnevale & Young (1989, pp 147–148) show that the modified dynamics will lead to at least one non-trivial stationary, generally stable, solution of the Euler equations of fluid motion from any initial conditions. This is shown as follows:

In two-dimensions consider the case where α is chosen so that energy increases. For all time, the enstrophy is still conserved. That is

$$\begin{aligned} Q &= \frac{1}{2}\int_{\mathcal{D}} q^2 dA \\ &= \frac{1}{2}\int_{\mathcal{D}} (\nabla^2\Psi)^2 dA \end{aligned} \tag{36}$$

is invariant, where $\mathcal{D}$ is our domain, and where Ψ is a stream function such that $\boldsymbol{u} = (-\Psi_y, \Psi_x)$. The energy of the flow is given by

$$\begin{aligned} E &= \frac{1}{2}\int_{\mathcal{D}} u^2 dA \\ &= \frac{1}{2}\int_{\mathcal{D}} (\nabla\Psi)^2 dA \end{aligned} \tag{37}$$

Although the energy monotonically increases until a solution is reached, its value is bounded from above by Poincaré's inequality

$$\int (\nabla^2\Psi)^2 \;\geq\; C\int (\nabla\Psi)^2 dA \tag{38}$$

where C is some constant and $\nabla\Psi$ vanishes on the boundary of the integral.

The energy of the flow is bounded above, so the process of energy growth must eventually stop. Thus for any two-dimensional flow, there exist at least one stationary solution of the Euler equations accessible by rearrangement of the vorticity field (while maintaining its initial topology/coadjoint orbit!) Excepting special cases, this state will be stable.

The above proof does not show that every stable solution on a given isovortical sheet (coadjoint orbit) may be found, even if there exists more than one. Also, the method may take an infinite time to reach a stable solution, especially if the solution has different topological properties. In that case vorticity reconnection can occur (See Figure 5)

Carnevale & Vallis (1990) extend the above result for two-dimensional flows and show that there can be at most two Arnold stable states per isovortical sheet (Carnevale & Vallis 1990, pp 553–554). The proof shows that there must exist at most two stable states which satisfy Arnold's convexity estimates. The convexity estimates are constructed using the energy of the flow and the generalized enstrophy

$$Q_F[\Psi] = \int F(q) dA$$

which is conserved for any function F.

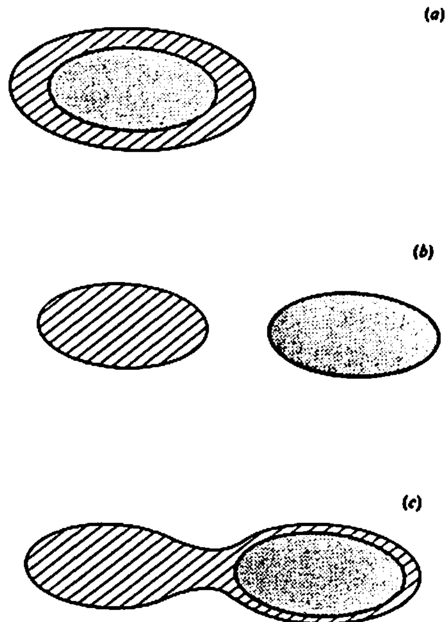

FIGURE 5. Schematic evolution of vortex patches. The areas of the patches remain conserved. (a) Initial state, (b) forbidden (in finite time) later state, (c) allowed later state.

Figure 5: Vortex Reconnection

The existence proof of Carnevale & Vallis (1990) does not preclude the existence of other stable states. In particular, it does not preclude the existence of nonlinearly stable states, which do not satisfy Arnold's rather strict convexity criteria.

Carnevale & Vallis (1990, p 55) note that there is a link between their result an a generalization of Andrews (1984) theorem. Andrews's theorem states that solutions satisfying Arnold (1966) stability criteria must have the same symmetries as the physical specifications (boundary conditions, topography, Coriolis parameter) of the problem. Because of the uniqueness arguments given in the proof in Carnevale & Vallis (1990), a state satisfying Arnold's criteria cannot be used to create a family of equal-energy solutions by shifts in the direction of symmetry of the problem. Thus, the stable solutions must have the same symmetry of the problem.

In the case of flow over topography, the Arnold stable states have the same symmetries as the topography. For situations with no topography there is nothing to fix the phase, except perhaps other invariants which cannot be expressed as functionals of the vorticity. There may be a continuum of nonlinearly stable equal-energy states with the members differentiated by the specification of another invariant not accounted for in the Arnold stability proof (Carnevale & Vallis 1990, p 555). For further discussion on this topic, the reader is referred to Chern & Marsden (1990).

It is stated that Kelvin ran the modified dynamics equilibrium search algorithm as a thought experiment over 100 years ago. It is interesting to note that his thought experiment is being actively carried out by contemporary researchers using computer computations.

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