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Additional Topics in Differential Geometry with an Aim Toward Further Studies of the General Theory of Relativity

Introduction:

To study the general theory of relativity it is necessary first to understand the mathematical formalism in which the theory is cast. This paper aims to bridge the gap between the concepts which were discussed in class and those which are introduced in Hawking and Ellis' The large scale structure of space-time. In their formulation of the mathematical theory of general relativity, Hawking and Ellis begin by defining the structure of space-time; i.e., a connected four-dimensional C^∞ Hausdorff manifold M with a Lorentz metric g. They also introduce a number of other concepts from differential geometry which are important parts of the mathematical formalism --vectors, tensors, covariant derivatives, curvature tensors, etc. I review here only those major concepts which are new or which are discussed in significantly more detail than in Mechanics and Symmetry.

Tensors:

From the tangent and cotangent spaces to a manifold at a point p, form the Cartesian product

$$\Pi^s_r = T^*_pM \times \ldots \times T^*_pM \times T_pM \times \ldots \times T_pM.$$

A ***tensor of type (r,s)*** at p is a function on Π^s_r which is linear in each argument. The space of all tensors of type (r,s) is denoted

$$T^r_s(p) = T_p \otimes \ldots \otimes T_p \otimes T^*_p \otimes \ldots \otimes T^*_p.$$

Note that, $T^1_0(p) = T_p$ and $T^0_1(p) = T^*_p$.

If R and S are tensors at a point p of type (r,s) and (p,q) respectively, then the ***tensor product*** $R \otimes S$ is an (r+p,s+q) tensor at p defined by

$$R \otimes S(\alpha^1,\ldots,\alpha^{r+p},X_1,\ldots,X_{s+q}) = R(\alpha^1,\ldots,\alpha^r,X_1,\ldots,X_s)S(\alpha^{r+1},\ldots,\alpha^{r+p},X_{s+1},\ldots,X_{s+q}).$$

The components of a tensor T of type (r,s) at p with respect to a pair of dual bases $\{E_a\}$, $\{E^a\}$ of the tangent and cotangent spaces at p are given by

$$T^{a_1\ldots a_r}{}_{b_1\ldots b_s} = T(E^{a_1},\ldots,E^{a_r},E_{b_1},\ldots,E_{b_s}),$$

so

$$T = T^{a_1 \cdots a_r}{}_{b_1 \ldots b_s} E^{a_1} \otimes \cdots \otimes E^{a_r} \otimes E_{b_1} \otimes \cdots \otimes E_{b_s}.$$

The upper and lower indices of T are called contravariant and covariant indices respectively. In particular, a (1,0) tensor is said to be a ***contravariant vector***, and a (0,1) tensor (or one-form) is said to be a ***covariant vector***.

The ***contraction*** of a tensor T of type (r,s), with components $T^{a_1 \cdots a_r}{}_{b_1 \ldots b_s}$ with respect to a pair of dual bases, on the ith contravariant and jth covariant indices is defined to be the tensor $C^i_j(T)$ of type (r-1,s-1) whose components with respect to the same bases are $T^{a_1 \ldots a_{i-1} a_i a_{i+1} \ldots a_r}{}_{b_1 \ldots b_{j-1} a_i b_{j+1} \ldots b_s}$, i.e.

$$C^i_j(T) = T^{a_1 \ldots a_{i-1} a_i a_{i+1} \ldots a_r}{}_{b_1 \ldots b_{j-1} a_i b_{j+1} \ldots b_s} E_{a_1} \otimes \cdots \otimes E_{a_{i-1}} \otimes E_{a_{i+1}} \otimes \cdots \otimes E_{a_r} \otimes E_{b_1} \otimes \cdots \otimes E_{b_{j-1}} \otimes E_{b_{j+1}} \otimes \cdots \otimes E_{b_s}$$

The symmetric part of a tensor T of type (k,0) is the tensor S(T) defined by

$$S(T)^{a_1 \cdots a_k} = \frac{1}{k!} \sum_{\sigma \in S_k} T^{a_{\sigma(1)} \cdots a_{\sigma(k)}}$$

The components of this tensor are denoted $T^{(a_1 \cdots a_k)}$. Similarly, the components of the anti-symmetric part of T are denoted by

$$T^{[a_1 \ldots a_k]} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T^{a_{\sigma(1)} \cdots a_{\sigma(k)}},$$

where S_k is the group of permutations and sgn(σ) is the sign of the permutation σ. More generally, the components of the symmetric or antisymmetric part of a tensor on a particular set of covariant or contravariant indices can be denoted by placing round or square brackets around the indices. E.g., $T_{(a_1 \cdots a_r)}{}^{b_1 \cdots b_s}$ or $T_{[a_1 \cdots a_r]}{}^{b_1 \cdots b_s}$.

A tensor is ***symmetric*** with respect to a given set of indices if it is equal to its symmetric part on these indices. It is ***antisymmetric*** with respect to a given set of indices if it is equal to its antisymmetric part on these indices.

A ***q-form*** is a tensor of type (0,q) that is antisymmetric on all q indices. In this notation, the wedge product of a q-form A with a p-form B is defined by

$$(A \wedge B)_{a_1 \cdots a_{q+p}} = A_{[a_1 \cdots a_q} B_{a_{q+1} \cdots a_{q+p}]}$$

A ***C^k tensor field*** T of type (r,s) on a set U in M is an assignment of an element of $T^r_s(p)$ to each point p in U such that the components of T with respect to any coordinate basis defined on an open subset of U are C^k functions. Note that a vector field is a tensor field of type (1,0).

If a set of local coordinates are chosen on some open set in M, and if basis vectors $\{(\partial / \partial x^a)_p\} \subset T_pM$ and $\{(dx^a)_p\} \subset T^*_pM$ can be chosen, then this basis is called a

coordinate basis. There is one advantage to using such a basis. That is that $E_a(E_b f) = E_b(E_a f)$ (by the equality of mixed partials).

The Covariant Derivative:

Unlike the exterior and Lie derivatives, which can be defined independently of any additional structure on the manifold, the covariant derivative requires the addition of an new structure called a connection. (The following definition of the covariant derivative is equivalent to that given in Mechanics and Symmetry, but as this construction generalizes more easily to the definition of the covariant derivative for arbitrary tensors I will outline it here.)

A ***connection*** ∇ at a point p of M is a rule which takes each vector field X at p and assigns an operator ∇_X which maps each vector field Y (Y is at least C^1) to a vector field $\nabla_X Y$ such that:

(i) for any functions f and g on M, and C^1 vector fields X, Y, Z

$$\nabla_{fX+gY} Z = f\,\nabla_X Z + g\nabla_Y Z$$

(ii) for any C^1 vector fields Y, Z and real numbers c, d

$$\nabla_X(cY+dZ) = c\nabla_X Y + d\nabla_X Z$$

(iii) for an C^1 function F and C^1 vector field Y

$$\nabla_X(fY) = X(f)Y + f\,\nabla_X Y$$

$\nabla_X Y$ is the ***covariant derivative*** (with respect to ∇) of Y ***in the direction*** X at p. By (i) we have that $\nabla_X Y = \nabla_{X^a E_a} Y = X^a \nabla_{E^a} Y$, which implies that there exists a tensor of type (1,1) dependent only on Y which gives the vector $\nabla_X Y$ when contracted with the vector X. This tensor, denoted by ∇Y, is called the ***covariant derivative*** of Y. Locally, the components of ∇Y are written $Y^a{}_{;b}$ so

$$\nabla Y = Y^a{}_{;b} E^b \otimes E_a$$

Condition (iii) above is equivalent to $\nabla(fY) = df \otimes Y + f\,\nabla Y$.

The connection is then determined by the functions $\Gamma^a{}_{bc}$, where

$$\Gamma^a{}_{bc} = \left\langle E^a, \nabla_{E_b} E_c \right\rangle = \left\langle E^a, \delta^d{}_b (E_c)^e{}_{;d} E_e \right\rangle = (E_c)^a{}_{;b} \Leftrightarrow \nabla E_c = \Gamma^a{}_{bc} E^b \otimes E_a.$$

Now we can see that

$$\nabla Y = \nabla\left(Y^c E_c\right) = dY^c \otimes E_c + Y^c \otimes \nabla E_c = dY^c \otimes E_c + Y^c \Gamma^a{}_{bc} E^b \otimes E_a.$$

And so the components of ∇Y and $\nabla_X Y$ with respect to the coordinate bases are

$$Y^a{}_{;b} = \partial Y^a / \partial x^b + \Gamma^a{}_{bc} Y^c$$

and

$$(\nabla_X Y)^a = X^b Y^c \Gamma^a{}_{bc} + X^b \partial Y^a / \partial x^b.$$

The last of these equations is equivalent to (7.5.2) in Mechanics and Symmetry. Placing $\gamma(X,Y) = X^b Y^c \Gamma^a{}_{bc} E_a$ completes the comparison between the two discussions. In particular, the concept of ***parallel transport*** as it is defined in Mechanics and Symmetry is equivalent to the definition which is given in (Ellis and Hawking, p.32), except that the definition given there is generalized to arbitrary tensors on M.

The second covariant derivative $\nabla\nabla Z$ of a vector Z is defined to be the covariant derivative of ∇Z. Its components are $Z^a{}_{;bc}$.

The covariant derivative can also be generalized. The covariant derivative of any C^r tensor field T is defined by the following rules:

(i) if T is of type (q,s), then ∇T is a C^{r-1} tensor field of type (q,s+1)
(ii) ∇ is linear and commutes with contractions
(iii) for any tensor fields S and T, $\nabla(S\otimes T) = (\nabla S)\otimes T + S\otimes(\nabla T)$
(i.e., ∇ is a derivation)
(iv) $\nabla f = df$

The components of ∇T are written $(\nabla_{E_h} T)^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s ;h}$.

The following relation relating the Lie derivative to the covariant derivative will be useful:

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

The Riemann Curvature Tensor:

In general, the second covariant derivative is non-commutative, i.e., $Z^a{}_{;bc} \neq Z^a{}_{;cb}$. The ***Riemann curvature tensor*** R gives an expression for the non-commutation of $\nabla\nabla Z$.

Given C^{r+1} vector fields X, Y, Z, one can use the C^r connection ∇ to define a C^{r-1} vector field R(X,Y)Z by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z.$$

Then with $R^a{}_{bcd} := \langle E^a, R(E_c, E_d) E_b \rangle$, we find

$$\begin{aligned}(R(X,Y)Z)^a &= R^a{}_{bcd} X^c Y^d Z^b = (Z^a{}_{;d} Y^d)_{;c} X^c - (Z^a{}_{;d} X^d)_{;c} Y^c - Z^a{}_{;d}(Y^d{}_{;c} X^c - X^d{}_{;c} Y^c) \\ &= Z^a{}_{;dc} Y^d X^c + Z^a{}_{;d} Y^d{}_{;c} X^c - Z^a{}_{;dc} X^d Y^c - Z^a{}_{;d} X^d{}_{;c} Y^c - Z^a{}_{;d}(Y^d{}_{;c} X^c - X^d{}_{;c} Y^c) \\ &= (Z^a{}_{;dc} - Z^a{}_{;cd}) X^c Y^d\end{aligned}$$

which is equivalent to $Z^a{}_{;dc} - Z^a{}_{;cd} = R^a{}_{bcd} Z^b$ since X and Y are arbitrary vectors.

In coordinate bases,

$$R^a{}_{bcd} = \partial\Gamma^a{}_{db} / \partial x^c - \partial\Gamma^a{}_{cb} / \partial x^d + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb}$$

If $R^a{}_{bcd} = 0$ at all points of M, then the connection is said to be ***flat***.

Furthermore, the Riemann tensor has the following properties (as can be verified from the definition):

$$R^a{}_{b(cd)} = 0 \Leftrightarrow R^a{}_{bcd} = R^a{}_{bdc}$$

$$R^a{}_{[bcd]} = 0 \Leftrightarrow R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0$$

$$R^a{}_{b[cd;e]} = 0 \Leftrightarrow R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0$$

Using the metric tensor g (discussed later), and putting $R_{abcd} = g_{ae}R^e{}_{bcd}$, one derives two further properties of the Riemann tensor:

$$R_{(ab)cd} = 0 \Leftrightarrow R_{abcd} = -R_{bacd}$$

$$R_{abcd} = R_{cdab}$$

The last of these implies the ***Ricci tensor*** of type (0,2) (constructed by contracting the curvature tensor) with components $R_{bd} = R^a{}_{bad}$ is symmetric.

The Metric

A ***metric tensor*** g at a point p in M is a symmetric tensor of type (0,2) at p. A C^r metric on M is a C^r symmetric tensor field g. With the metric tensor it is possible to define the length of a vector as well as the 'cos angle' between two non-zero vectors.

The 'magnitude' of a vector X is given by $(|g(X,X)|)^{1/2}$ and the 'cos angle' between two vectors X and Y is given by $g(X,Y)/(|g(X,X)\, g(Y,Y)|)^{1/2}$, where $g(X,X)\, g(Y,Y) \neq 0$. X and Y are said to be ***orthogonal*** if $g(X,Y) = 0$.

The coordinates of g with respect to a basis $\{E_a\}$ are given by $g_{ab} = g(E_a,E_b)$.

Since g is assumed to be a non-degenerate metric (i.e., $g(X,Y) = 0$ for all vectors Y <=> X=0), the matrix (g_{ab}) is non-singular and inevitable. There therefore exists a unique tensor of type (2,0) with components g^{ab}, where $(g^{ab}) = (g_{ab})^{-1}$. g^{ab} is defined by

$$g^{ab}\, g_{bc} = \delta^a{}_c.$$

The ***signature*** of g at p is defined to be the number of positive eigenvalues of (g_{ab}), minus the number of negative ones. A metric is called a ***Lorentz metric*** on M if it has signature n-2. The existence of a Lorentz metric allows the division of the nonzero vectors at p into three classes ***--timelike, null or spacelike--*** each characterized by the sign of g(X,X) --negative, zero or positive, respectively.

There is also a relation between the metric and the connection which was discussed earlier. For the metric g (in fact for any metric) there is a unique connection defined by the condition that the covariant derivative of g is zero, i.e. $g_{ab;c} = 0$. One can then derive from this condition expressions for the connection components. (Ellis and Hawking, p.

40) One finds that if a coordinate basis is chosen that the connection components are given by the Christoffel relations (i.e. equation (7.5.14) in Mechanics and Symmetry).

Conclusion:

This completes the review of the major points of difference between the concepts discussed in class and those in the chapter on differential geometry in The large scale structure of space-time. Among the points which were not discussed were the generalizations of the exterior and Lie derivatives to arbitrary tensors, as well as the section on hypersurfaces. Also not discussed, but of particular importance was the notion of a Killing vector, which will evidently play a role in the derivation of the ***Einstein equations*** (Ellis and Hawking, p. 74).

References:

Ellis, G.F.R. and Hawking, S.W. (1991), *The large scale structure of space-time*. (Cambridge University Press, Cambridge).