Sophus Lie and Symmetry

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In high school I have to admit that I found mathematics both boring and difficult. The inane complexities of calculus, I thought, jabbing a pencil deep into my hated math textbook.

Times change. I no longer find mathematics boring, though it is certainty still difficult. As a description of the universe, mathematics rewards clarity, encompasses complexity, and preserves subtlety. A high art indeed, but its ways are not always so simple to learn. To a rank apprentice, standing at the bottom of such an edifice of knowledge, there are many questions and many difficulties, such as:

- Algebra
- Linear algebra
- Analysis
- Geometry
- They are all the same, they are all different. How exactly do they relate? What can we build in and amongst and on top of them?
- Will I ever understand ANY of these subjects even a little bit?

I have not come upon the answers to these questions, but I have come upon some very interesting things nonetheless. In these few pages I would like to describe some of the things that I have found (most of them lying about the math library) that seem to me to be especially intriguing.

Physics, like mathematics, was created towards a better understanding of nature. To me it seems that these two fields should be basically one and the same, but that is apparently not how the world works. I think it has always been more or less true that the language of physics is mathematics, however. Whatever their true relationship, I think that the connections between these two paradigms? languages? disciplines? represent some of the most beautiful acheivements of human culture. There are three intertwined threads that I would like to point out in particular, these being mechanics, geometry, and the theory of Lie groups.

In the study of mechanics, as in the rest of physics, we are looking for patterns in natural systems. These patterns sometimes come to be called laws. Sometimes they remain only theories, but they are valuable anyway if they reveal. By reveal I mean that they open even a small doorway into the universe through which we can send our minds for understanding. Our minds understand patterns: "Ah, as it is there so is it here." And so we look for them.

There is an old saying to the effect that we find only what we are ready to find, and we see only what we are ready to see. Physics and mathematics are essentially descriptive; to describe something new, the language must be expanded and enriched.

Geometry and mechanics have a history together. Hamilton initiated the study of the phase space of a mechanical system as a key to understanding the system itself. The geometry of phase space is called symplectic geometry, and is perhaps the essence of the modern conception of mechanics (see Guillemin and Sternberg[1984], and Marsden and Ratiu[1993]). The relationships between mechanics and the theory of Lie groups follow a tortuous path through history, with wheels being constantly reinvented in different places, times, and guises. Consider the following quote taken from Weinstein [1983].

> ...many results in symplectic geometry usually attributed to various mathematicians during the 1960's can be found in Sophus Lie's book *Theorie der Transformationsgruppen.*

Many of these patterns that we look for can also be called symmetries. In mathematics, the description of symmetry involves the use of structures, the canonical example being the *group*.The contribution of Sophus Lie was to advance our understanding of the nature of symmetry, and thus he helped set the stage for the fundamental advances of this century's physics and mathmatics.

One way Lie's work has been described is as the study of the relationships between groups and systems of differential equations (see Gilmore [1974] and Hermann [1975]). Here there is a link with another mathematician, E. Noether, who formulated a fundamental statement about symmetries present in systems of differential equations.

Noether's famed theorem is thus described in Olver[1993]:

> Noether's theorem provides a connection between one-parameter variational symmetry groups of the system and conservation laws or first integrals.

Olver further goes on to say that when one has such an integral, one can reduce the order of the system and so are on the way toward integrating the system by quadratures.

Already this is all mixed up with Lie's ideas. Perhaps I should go a little more slowly.

First of all, a symmetry group of a system is a group acting on the variables of said system such that it takes solutions of the system to other solutions. Apparently Lie knew something about these matters as well, since Olver tells us that Lie showed how finding a one-parameter symmetry group leads to a reduction by one of the order of the system. Anyway, it seems clear that Lie was actively studying systems of partial differential equations (see also Gilmore[1974]) which led him to the study of continous transformation groups, from which crucible we have the theory of Lie groups.

A Lie group at its most basic level is just a continuous group. For Lie himself, the basic object was a Lie transformation group, which is a Lie group and a manifold, together with a group action under some conditions. This same concept, with maybe slightly different conditions on the group action, is also called the representation of the group on the manifold. One more term: a Lie algebra is a vector space with a bilinear operation.

So now we can wonder what Lie did with these objects, and what we can do with these objects today.

Before I move off in a somewhat unrelated direction, let us return briefly to the matter of Noether's theorem and what Lie and his theory may have to do with systems of differential equations.

There is no doubt that Lie was motivated by and interested in the theory of differential equations. He considered himself in the tradition of Abel and Galois, in that he did for differential equations what they did for algebraic equations. It is interesting to note from Lie's introduction to *Theorie der Transformationsgruppen* that he considered his work to be based in both geometry and analysis, each supporting the other, especially when one considers the interrelationships between geometry, the theory of differential equations, analysis, and mechanics as these fields have developed.

This trend that began with Lie himself continues. Lie groups appear as symmetry groups in a wide range of physical systems of interest. For example, consider the following quote from Marsden[1969].

> In general, from Lie group actions which leave Hamiltonians fixed, we hope to extract conserved functions (also called integrals).

So Lie groups are an important tool for obtaining conservation laws of Hamiltonian systems in particular.

Here is another suggestive quote regarding modern application of Lie groups to systems of differential equations (from Gilmore[1974]).

> Lie groups have been studied so extensively iin their own right that their connection with partial differential equations is often overlooked and forgotten. So it is sometimes quite a shock to learn that many of the differential equations of mathematical physics are expressions of the Casimir invariant of some Lie group in a particular representation and, moreover, that all the standard special functions of mathematical physics are simply related to matrix elements in the representations of a few of the simplest Lie groups.

Apart from considering Lie groups as symmetry groups, there is another tack that Lie took, and that recent developments have taken, relating Lie's ideas directly to the consideration of Hamiltonian systems.

The first important tool that is needed Lie provided in his three fundamental theorems concerning the relationship between Lie groups and Lie algebras. He showed that for each Lie group there is a corresponding Lie algebra, and characterized the resulting Lie algebra via structure constants (via a left-extension, to use another term).

The next step is to note that many crucial mechanical systems have a Lie group G as a configuration space. Following Marsden[1982], a few examples:

- the free rigid body system is associated to $SO(3)$ =rotation group
- the perfect incompressible fluid system is associated to $\mathcal{D}_{vol}$ = volume preserving diffeomorphisms
- the Heisenberg equation of quantum mechanics is associated to $\mathcal{U}(\mathcal{H})$, unitary group of complex Hilbert space
- the Poisson-Vlasov equations of plasma physics are associated to $\mathcal{L}$ = group of canonical transformations.

Now consider the dual of the corresponding Lie algebra, $\mathcal{G}^*$. Lie defined what is now called a Poisson structure on $\mathcal{G}^*$ by

$$\{F,G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle$$

where $F, G : \mathcal{G}^* \longrightarrow \mathbf{R}$ are smooth, real valued functions. This bracket is called the Lie-poisson bracket. $\mathcal{G}^*$ and its bracket form a Poisson manifold (this is the modern terminology, but Lie apparently developed the concept; c.f. Marsden[1993]).

Next, follow the standard procedure (given a bracket) to arrive at the equations of motion for the system. Given $H : \mathcal{G}^* \longrightarrow \mathbf{R}$ an energy function, there is a unique vector field X_H on $\mathcal{G}^*$ defined by the equation

$$DF(\mu) \cdot X_H(\mu) = \{F, H\},$$

or in other words $\frac{dF}{dt} = \{F, H\}$, the evolution equations determined by H.

A system thus described leaves the coadjoint orbits in $\mathcal{G}^*$ invariant.

This procedure is just one example of the development of Lie's ideas in modern mathematics, but for me it is very representative of the descriptive power of the structures Lie explored. He laid much of the foundation for modern mechanics, in that by using the language of Lie groups we are able to very succinctly use tools from geometry, algebra, and analyis to understand mechanical systems.

I believe that it is just this unity of form and function that mathematics and physics, at their best, aim towards. The theories of Sophus Lie mark rich ground to explore.

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