Francesco - An excellent & sophisticated report! We should continue to discuss the area we started on "controlling using gyroscopic forces".

INVARIANT AFFINE CONNECTIONS AND CONTROLLABILITY ON LIE GROUPS

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Final Project for CIT-CDS 141a
Geometric Mechanics, Fall 1995
December 8, 1995

ABSTRACT. This report focuses on invariant affine connections on Lie groups and on their applications to nonlinear control theory. After a brief review of Riemannian geometry, we introduce invariant connections and use them to rederive the Euler-Poincaré equations. We then apply the same reduction technique to the Jacobi equations for the linearization of the Lagrange's equations. Finally we consider the controllability problem for simple mechanical systems on Lie groups. The theory developed in [LM95] leads naturally to a simple algebraic test.

1. INTRODUCTION

This report underlines the strong interaction between the areas of geometric mechanics and nonlinear control. We employ tools from Riemannian geometry and Lie group theory to study the controllability properties of invariant mechanical systems within the framework recently developed in [LM95]. In particular, we adopt the notion invariant affine connection to study configuration controllability and to obtain an algebraic test for the class of group invariant systems: notable examples are the rigid body with external torques and the forced planar rigid body. Standard results present in the literature are recovered through algebraic computations.

We apply the theory to these instructive examples with the hope of providing a viable tool in developing intuition as well as motivating further theoretical analysis. Also, by studying physical examples, we keep our attention focused on problems of concrete relevance.

The report is organized into four sections: the first three sections contain first a review of well-known or well-documented general results and then they analyse in some detail the Lie group case. In Section 2 we introduce the notion of affine connections and invariant affine connections on a Lie group. In Section 3 we present Lagrange's equations and show an alternative way to derive the Euler-Poincaré equations. We then go on and apply

Lagrange's reduction to the linearization of Lagrange's equations to obtain the so-called reduced Jacobi equation. In Section 4 we review the controllability definitions and tests described in [LM95] and then, once again, specialize the results to group invariant systems. Finally, Section 5 contains two relevant examples: the forced planar rigid body and the rigid body with external torques. Describing in a table what we have just said:

	manifold M	*Lie group* G
Sec. 2	Riemannian connection	invariant Riemannian connection
Sec. 3	Lagrange's equations and Jacobi equation	Euler-Poincaré equations and reduced Jacobi equation
Sec. 4	symmetric product as map from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$	symmetric product as map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$
Sec. 5	planar rigid body and rigid body examples.	

2. Invariant Riemannian connections on Lie groups

For a standard introduction to Riemannian geometry we refer to [Car92] and [AM87, Section 2.7]. Regarding the notion of left-invariant connections on a Lie group, we follow the treatment in [Hel78, Section II.3] and [Arn89, Appendix B].

Definition 1. A *Riemannian metric* on manifold M is a tensor $g \in \mathfrak{T}^0_2(M)$ such that for all $p \in M$, g_p is a symmetric positive-definite bilinear form on T_pM.

We write g in local coordinates as g_{ij} and we sometimes denote g_p with the symbol $\langle\langle \cdot, \cdot \rangle\rangle_p$. Additionally, we denote with the symbols $^\flat : TM \to T^*M$ and $^\sharp : T^*M \to TM$ the musical isomorphisms associated with g.

Definition 2. An *affine connection* on M is a map that assigns to each pair of smooth vector fields X, Y a smooth vector field $\nabla_X Y$ such that

i) $\nabla_{fX} Y = f \nabla_X Y$ and
ii) $\nabla_X fY = f \nabla_X Y + \mathcal{L}_X f \, Y$

for all $f \in C^\infty(M)$. In a local chart with coordinates (x^i) we define the *Christoffel symbols* by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}. \tag{2.1}$$

Given any three vector fields X, Y, Z on M, we say that the affine connection ∇ on M is *torsion-free* if

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

and is *compatible* with the metric g if

$$\mathcal{L}_X \langle\langle Y, Z \rangle\rangle = \langle\langle \nabla_X Y, Z \rangle\rangle + \langle\langle Y, \nabla_X Z \rangle\rangle.$$

Theorem 1 (Levi-Civita). *Given a Riemannian manifold M, there exists a unique torsion-free affine connection ∇ on M compatible with the metric. We call this ∇ the Riemannian (Levi-Civita) connection; it satisfies*

$$2\langle\langle X, \nabla_Z Y \rangle\rangle = \mathcal{L}_Z \langle\langle X, Y \rangle\rangle + \langle\langle Z, [X, Y] \rangle\rangle + \mathcal{L}_Y \langle\langle X, Z \rangle\rangle + \langle\langle Y, [X, Z] \rangle\rangle - \mathcal{L}_X \langle\langle Y, Z \rangle\rangle - \langle\langle X, [Y, Z] \rangle\rangle. \tag{2.2}$$

Equation (2.2) allows us to compute the Christoffel symbols. A tedious computation shows that

$$\Gamma^k_{ij} = \frac{1}{2} g^{hk} \left\{ \frac{\partial g_{hj}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^h} \right\}.$$

Finally, let the *curvature tensor* $R \in \mathfrak{T}^1_3(M)$ be

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and, given a two-dimentional subspace $\Gamma_p \subset T_pM$ of the tangent space T_pM and an orthonormal basis $\{X_p, Y_p\}$ of Γ_p, let the *sectional curvature* of Γ_p be

$$K(\Gamma_p) = \langle\!\langle X_p, R(X_p, Y_p, Y_p)\rangle\!\rangle,$$

where well-posedness can be shown. □

We now specialize these results to Lie groups. A Riemannian metric on a Lie group G is called *left-invariant* if it is preserved by all left translations L_g. Therefore, the metric is uniquely determined by its value at the identity, which is an inner product on the Lie algebra $\mathfrak{g}$.

Definition 3. An affine connection ∇ is said to be *left-invariant* if

$$(\nabla_{\tilde{X}}\tilde{Y})_g = T_eL_g(\nabla_{\tilde{X}}\tilde{Y})_e$$

for each pair of left-invariant vector fields $\tilde{X}, \tilde{Y}$ on G.

Theorem 2. *The Riemannian connection of a left-invariant Riemannian metric is also left-invariant. We denote by $\overline{\nabla} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ its restriction to the identity and we call it the* **reduced connection.**

For all $X, Y \in \mathfrak{g}$, we have

$$\overline{\nabla}_X Y = \frac{1}{2}[X,Y]_{\mathfrak{g}} - \frac{1}{2}\left(\mathrm{ad}_X^* Y^\flat + \mathrm{ad}_Y^* X^\flat\right)^\sharp, \tag{2.3}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie bracket on $\mathfrak{g}$, $\mathrm{ad}_X Y = [X,Y]_{\mathfrak{g}}$ and ad_X^* is the dual operator of ad_X on $\mathfrak{g}^*$.

Proof. By Theorem 1, there exists a unique torsion-free affine connection ∇ compatible with the left-invariant metric. Equation (2.2) allows us to compute an explicit expression for it. Denote with $\tilde{X}, \tilde{Y}, \tilde{Z}$ left-invariant vector fields on G and with $X, Y, Z \in \mathfrak{g}$ their value at the identity T_eG. Denote with $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}}$ the inner product on $\mathfrak{g}$ and with $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ its left translation to TG. We have

$$2\langle\!\langle\tilde{Z}, \nabla_{\tilde{X}}\tilde{Y}\rangle\!\rangle = \langle\!\langle\tilde{X}, [\tilde{Z},\tilde{Y}]\rangle\!\rangle + \langle\!\langle\tilde{Y}, [\tilde{Z},\tilde{X}]\rangle\!\rangle - \langle\!\langle\tilde{Z}, [\tilde{Y},\tilde{X}]\rangle\!\rangle$$

since, for example, $\langle\!\langle\tilde{X},\tilde{Y}\rangle\!\rangle = \langle\!\langle X, Y\rangle\!\rangle_{\mathfrak{g}}$ is constant and $\mathcal{L}_{\tilde{X}}\langle\!\langle\tilde{X},\tilde{Y}\rangle\!\rangle$ vanishes. We can now pull back the equation to the identity to obtain

$$2\langle\!\langle Z,\ T_gL_{g^{-1}}(\nabla_{\tilde{X}}\tilde{Y})_g\rangle\!\rangle_{\mathfrak{g}} = \langle\!\langle X, [Z,Y]\rangle\!\rangle_{\mathfrak{g}} + \langle\!\langle Y, [Z,X]\rangle\!\rangle_{\mathfrak{g}} - \langle\!\langle Z, [Y,X]\rangle\!\rangle_{\mathfrak{g}}. \tag{2.4}$$

We can evaluate the left hand side of this equation at a generic g and at the identity e; since the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}}$ is non-degenerate, we have

$$T_gL_{g^{-1}}(\nabla_{\tilde{X}}\tilde{Y})_g = (\nabla_{\tilde{X}}\tilde{Y})_e.$$

Therefore the connection ∇ is left-invariant. Equation (2.3) also follows from equation (2.4) using the non-degeneracy of $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}}$. □

Remark 1 (Symmetric product). On the Lie algebra $\mathfrak{g}$, note the decomposition of the covariant derivative into skew-symmetric and symmetric terms

$$2\overline{\nabla}_X Y = [X,Y]_{\mathfrak{g}} + \langle X : Y \rangle_{\mathfrak{g}},$$

where we call

$$\langle X : Y \rangle_{\mathfrak{g}} \triangleq \overline{\nabla}_X Y + \overline{\nabla}_Y X \; = \; -(\operatorname{ad}_X^* Y^\flat + \operatorname{ad}_Y^* X^\flat)^\sharp$$

the *symmetric product* of X and Y. We shall see later the meaning of this definition in a control theoretic setting.

Remark 2 (Reduced curvature tensor). Also the curvature tensor R has an invariance property inherited from the affine connection ∇. It holds:

$$R(X_g^1, X_g^2, X_g^3) = T_e L_g \, R(x^1, x^2, x^3),$$

where $X_g^i \in T_g G$ and $x^i = T_g L_{g^{-1}} \, X_g^i \in \mathfrak{g}$.

We can therefore identify the curvature tensor R with its restriction to $T_e G = \mathfrak{g}$ and define the *reduced curvature* $\bar{R} \in \mathfrak{T}_3^1(\mathfrak{g})$ by

$$\bar{R}(X,Y,Z) = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]_{\mathfrak{g}}} Z.$$

where $X, Y, Z \in \mathfrak{g}$.

2.1. **Additional remarks.** Equation (2.3) is also present in Arnold [1] and will be rather useful for the controllability computations in the next sections. For now, we can use it to investigate some side issues:

Sectional curvature: Given a pair of independent vectors $X, Y \in \mathfrak{g}$, define $\bar{K}(\operatorname{span}\{X,Y\})$ to be the sectional curvature of the $\Gamma_g = \operatorname{span}\{\widetilde{X}, \widetilde{Y}\}_g$ (independence of g can be checked). For the case of orthonormal X and Y, we have

$$4\bar{K}(\operatorname{span}\{X,Y\}) = \|\langle X : Y \rangle\|^2 - 3\,\|[X,Y]\|^2 - \langle\!\langle (\operatorname{ad}_X^* X^\flat)^\sharp, (\operatorname{ad}_Y^* Y^\flat)^\sharp \rangle\!\rangle$$
$$+ 2\,\langle\!\langle [X,Y], (\operatorname{ad}_Y^* X^\flat - \operatorname{ad}_X^* Y^\flat)^\sharp \rangle\!\rangle,$$

where $\|X\|^2 = \langle\!\langle X, X \rangle\!\rangle$ and where the subscript $\mathfrak{g}$ is implicit everywhere.

Sectional curvature on $SO(3)$: Identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ through the natural Lie algebra isomorphism $\omega \mapsto \hat{\omega}$. Let $\mathbb{I} = \operatorname{diag}[J_1, J_2, J_3]$ be the matrix representation of g with respect to the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ (the letter $\mathbb{I}$ comes from inertia). Then, computing the previous expression,

$$\bar{K}(\operatorname{span}\{e_1, e_2\}) = \frac{(J_1 - J_2)^2 + J_3(2J_1 + 2J_2 - 3J_3)}{4 \det \mathbb{I}}.$$

As discussed in [Arn89, Appendix B], the sign of K (hence of $\bar{K}$) should affect the stability of geodesics on $SO(3)$ through the Jacobi equation:

[1] [Arn89, page 329], where the notation is $B(c,a) = (\operatorname{ad}_a^* c^\flat)^\sharp$.

in particular negative sectional curvature corresponds to instability of the geodesic flow and vice-versa.

Note that we are not discussing the stability of the Euler-Poincaré equations on the reduced space $\mathfrak{so}(3)$ (where an energy argument can be employed), but rather stability on the full configuration space $SO(3)$.

Bi-invariant connections: Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The $\langle\langle\cdot,\cdot\rangle\rangle$ metric on $\mathfrak{g}$, is said to be *bi-invariant* if

$$\langle\langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y\rangle\rangle = \langle\langle X, Y\rangle\rangle, \qquad \forall X, Y \in \mathfrak{g} \quad \text{and} \quad \forall g \in G.$$

or equivalently

$$\langle\langle X, [Y, Z]\rangle\rangle = \langle\langle [X, Y], Z\rangle\rangle, \qquad \forall X, Y, Z \in \mathfrak{g}.$$

For the case of bi-invariant metric, $\langle X : Y\rangle = -\left(\mathrm{ad}^*_X Y^\flat - \mathrm{ad}^*_Y X^\flat\right)^\sharp$ vanishes, so that

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \text{and} \quad R(X, Y, Z) = -\frac{1}{4}[X, [Y, Z]].$$

Also, it can be proven that the geodesics of the bi-invariant metric are the 1-parameter subgroups of G, that is curves of the form $\exp(Yt)$ for some fixed $Y \in \mathfrak{g}$. See [Car92, Exercises of Chapter 3 and 4] and [Hel78, Exercises of Chapter II].

3. Mechanical systems: Lagrange and Euler-Poincaré equations

This section describes how to write Lagrange's equations and mechanical control systems using the notion of affine connections introduced in the previous section, see [AM87]. As usual we deal with the Lie group case in the second part of the section, where the Euler-Poincaré equations are derived within the "invariant Riemannian connection" framework, see the elegant [SW86, Section 27, "Variations on a theme by Euler"] and the original work [Arn66] (in French).

Definition 4. A *simple mechanical control system* is defined by a Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on the configuration manifold Q (defining the kinetic energy), a function V on Q (defining the potential energy), and m one-forms, $F^1,\dots,F^m$, on Q (defining the inputs).

Let us denote with $q(t) \in Q$ the configuration of the system and with $\dot q(t) \in T_qQ$ its velocity. Using the formalism introduced in the previous section, Lagrange's equations for a simple mechanical control system can be written as

$$\nabla_{\dot q(t)}\dot q(t) = dV^\sharp(q(t)) + u^a(t)Y_a(q(t)), \tag{3.1}$$

where ∇ is the Riemannian connection associated with $\langle\langle\cdot,\cdot\rangle\rangle$ and $Y_a = (F^a)^\sharp$ are the input vector fields. (See the discussion in [AM87, Section 2.7] to correctly interpret the quantity $\nabla_{\dot q(t)}\dot q(t)$.)

□

Definition 5. A *simple mechanical control system on a Lie group* is defined by a left-invariant Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on the configuration group G (defining the kinetic energy), and m left-invariant one-forms, $F^1,\dots,F^m$, on G (defining the inputs).

Note that no non-trivial potential energy can be defined in a left-invariant fashion. Since the forms F^a are left-invariant, they are determined by their values at the identity $f^a = (F^a)_e \in \mathfrak{g}^*$. In particular we call $y_a = (f^a)^\sharp \in \mathfrak{g}$ the *input vectors.*

Theorem 3 (Euler-Poincaré equations). *Consider a simple mechanical control system on a Lie group. For a curve $g(t) \in G$, define a curve ξ in $\mathfrak{g}$ by $t \mapsto \xi(t) = T_{g(t)}L_{g(t)^{-1}}(\dot g(t))$. Then the following are equivalent:*

i) $g(t)$ satisfies Lagrange's equations (3.1) *on G;*

ii) **the Euler-Poincaré equations** *hold:*

$$\dot\xi = (\mathrm{ad}^*_\xi \xi^\flat)^\sharp + u^a y_a,$$

Note that if $g \in G$ is the system configuration, $\xi = g^{-1}\dot g \in \mathfrak{g}$ is the velocity in "body-frame".

Proof. We present a proof based on local coordinates and invariant connections. Let $\{X_1, \dots, X_n\}$ be a basis for $\mathfrak{g}$, then denote by $\{\widetilde{X}_1, \dots, \widetilde{X}_n\}$ the corresponding left-invariant basis for T_gG, that is $T_eL_g(X_k) = (\widetilde{X}_k)_g$ with the standard identification $T_eG = \mathfrak{g}$. If we decompose $\dot{g} = \xi^h \widetilde{X}_h$, then we recover

$$\dot{g} = T_eL_g(\xi), \quad \text{i.e.} \quad \dot{g} = g\,\xi,$$

where $\xi = \xi^h X_h \in \mathfrak{g}$.

Now, consider the left-invariant Riemannian connection ∇ associated with the invariant metric on G: Lagrange's equations (3.1) are written in terms of this ∇. Using the defining properties of an affine connection:

$$\begin{aligned}\nabla_{\dot{g}}\dot{g} &= \nabla_{\dot{g}}(\xi^k \widetilde{X}_k) \\ &= (\mathcal{L}_{\dot{g}}\xi^k)\widetilde{X}_k + \xi^k \nabla_{\dot{g}}\widetilde{X}_k \\ &= \frac{d\xi^k}{dt}\widetilde{X}_k + \xi^h \xi^k \nabla_{\widetilde{X}_h}\widetilde{X}_k\end{aligned}$$

since $\mathcal{L}_{\dot{g}}\xi^k$ is the directional derivative of ξ^k along $\dot{g}$, that is the time derivative.

Lagrangian reduction is performed by simply pulling back the previous equation to the identity

$$\begin{aligned}T_gL_{g^{-1}}(\nabla_{\dot{g}}\dot{g})_g &= \dot{\xi}^k X_k + \xi^h \xi^k \overline{\nabla}_{X_h} X_k \\ &= \dot{\xi} + \overline{\nabla}_{\xi}\xi,\end{aligned}$$

where, as in Theorem 2, we denote with $\overline{\nabla}$ the value of the invariant affine connection ∇ at the identity. The forced Euler-Poincaré equations are now recovered by recalling the expression of $\overline{\nabla}$ in equation (2.3) and the definition of y_a. □

3.1. **The reduced Jacobi equation.** In Riemannian geometry, a important relation between the two basic concepts of geodesics and curvature is given by the Jacobi equation. The latter determines the behavior of infinitesimal variations of a geodesic curve. Following the definitions in [Car92]:

Definition 6. Given a manifold M and a geodesic $\gamma : [a, b] \to M$. A vector field J along γ is said to be a *Jabobi field* if it satisfies the *Jacobi equation*

$$\nabla^2_{\dot{\gamma}(t)} J(t) + R(J(t), \dot{\gamma}(t), \dot{\gamma}(t)) = 0 \tag{3.2}$$

for all $t \in [a, b]$.

We now want to apply to the Jacobi equation the same technique we used to rederive Euler-Poincarè equations, i.e. reduction by left-translation to the identity.

Theorem 4 (Reduced Jacobi equation). *Let ∇ be a left-invariant Riemannian connection on G with $\overline{\nabla}$ the associated bilinear function on $\mathfrak{g}$. Let R be the curvature tensor on G and denote by $\bar{R}$ the associated tensor on $\mathfrak{g}$. For* *a geodesic g denote by $t \mapsto \xi(t) = T_{g(t)}L_{g(t)^{-1}}(\dot g(t))$ the associated solution of the Euler-Poincaré equations.*

For a vector field $t \mapsto J(t)$ along g, define a curve in $\mathfrak{g}$ by $t \mapsto \eta(t) = T_{g(t)}L_{g^{-1}(t)}(J(t))$. The following are equivalent:

i) J is a Jacobi field for g;

ii) $\eta(t) \in \mathfrak{g}$ satisfies the linear second-order equation

$$\ddot\eta + 2\overline{\nabla}_{\xi(t)}\dot\eta + \langle \xi(t) : [\xi(t), \eta]_{\mathfrak{g}}\rangle_{\mathfrak{g}} - [\overline{\nabla}_{\xi(t)}\xi(t), \eta]_{\mathfrak{g}} = 0\,; \tag{3.3}$$

we call this the **reduced Jacobi equation;**

iii) let $\sigma(t) \triangleq \dot\eta(t) + [\xi(t), \eta(t)]_{\mathfrak{g}}$. Then $(\eta, \sigma) \in \mathfrak{g} \times \mathfrak{g}$ satisfies the linear equation

$$\begin{aligned} \dot\eta &= -[\xi(t), \eta]_{\mathfrak{g}} + \sigma \\ \dot\sigma &= -\langle \xi(t) : \sigma\rangle_{\mathfrak{g}}. \end{aligned} \tag{3.4}$$

Proof. Write $J(t) = T_eL_{g(t)}\eta(t)$ and plug into equation (3.2). The left-invariance of ∇ leads to

$$\nabla_{\dot g(t)}J(t) = T_eL_{g(t)}\big(\dot\eta(t) + \overline{\nabla}_{\xi(t)}\eta(t)\big),$$

and iterating

$$\nabla^2_{\dot g(t)}J(t) = T_eL_{g(t)}\Big(\ddot\eta(t) + \overline{\nabla}_{\xi(t)}\dot\eta(t) + \frac{d}{dt}\overline{\nabla}_{\xi(t)}\eta(t) + \overline{\nabla}^2_{\xi(t)}\eta(t)\Big).$$

It is also possible to left-translate the curvature tensor R

$$R(J(t), \dot g(t), \dot g(t)) = T_eL_{g(t)}\bar R(\eta(t).\, \xi(t), \xi(t)),$$

where $\bar R$ is defined in Remark 2.

We can now substitute the two quantities into the Jacobi equation and perform Lagrangian reduction to obtain:

$$\ddot\eta(t) + 2\overline{\nabla}_{\xi(t)}\dot\eta(t) + \overline{\nabla}_{\dot\xi(t)}\eta(t) + \overline{\nabla}^2_{\xi(t)}\eta(t) + \bar R(\eta(t), \xi(t), \xi(t)) = 0.$$

This is a second order, linear ordinary differential equation in η. Considering only the term linear in η, we have:

$$\begin{aligned} \overline{\nabla}_{\dot\xi}\eta + \overline{\nabla}^2_{\xi}\eta + \bar R(\eta, \xi, \xi) &= \overline{\nabla}_{\dot\xi}\eta + \overline{\nabla}^2_{\xi}\eta + \overline{\nabla}_{\eta}\overline{\nabla}_{\xi}\xi - \overline{\nabla}_{\xi}\overline{\nabla}_{\eta}\xi - \overline{\nabla}_{[\eta,\xi]_{\mathfrak{g}}} \\ &= [\dot\xi, \eta]_{\mathfrak{g}} + \overline{\nabla}^2_{\xi}\eta - \overline{\nabla}_{\xi}\overline{\nabla}_{\eta}\xi - \overline{\nabla}_{[\eta,\xi]_{\mathfrak{g}}} \\ &= [\dot\xi, \eta]_{\mathfrak{g}} + \overline{\nabla}_{\xi}[\xi, \eta]_{\mathfrak{g}} - \overline{\nabla}_{[\eta,\xi]_{\mathfrak{g}}} \\ &= -[\overline{\nabla}_{\xi}\xi, \eta]_{\mathfrak{g}} + \langle \xi : [\xi, \eta]_{\mathfrak{g}}\rangle_{\mathfrak{g}} \end{aligned}$$

where we have used twice the fact that ∇ is torsion-free and once the definition of symmetric product on $\mathfrak{g}$.

Equation (3.4) follows after defining $\sigma = \dot\eta + [\xi, \eta]_{\mathfrak{g}}$. □

Remark 3 (Variational proof of the reduced Jacobi equation). We write Lagrange's equations for the simple mechanical control system on a Lie group in matrix notation as

$$\dot{g} = g\xi$$
$$\dot{\xi} = -\overline{\nabla}_\xi \xi = -\frac{1}{2}\langle \xi : \xi \rangle_{\mathfrak{g}}.$$

Let $(g(t), \xi(t)) \in G \times \mathfrak{g}$ be a solution to these equations.

Consider now a variation of the geodesic $g(t)$: $g \mapsto \delta g = g(t,s), s \in (-\epsilon, \epsilon)$, such that $g(t) = g(t, 0)$. Let

$$\eta(t,s) = g(t,s)^{-1} \frac{d}{ds}\Big|_{s=0} g(t,s) \quad \in \mathfrak{g}.$$

The variation of g induces a variation $\delta\xi$ of its angular velocity; call this variation $\sigma \in \mathfrak{g}$:

$$\sigma(t) = \delta\xi(t) = \frac{d}{ds}\Big|_{s=0} \xi(t,s),$$

where $\xi(t,s) = g(t,s)^{-1}\frac{d}{dt}\, g(t,s)$. Then the variations η and σ are related by the well-known relation

$$\frac{d}{dt}\eta = -[\xi, \eta]_{\mathfrak{g}} + \sigma.$$

This is the first of the two equations in system (3.4). The second equation is the linearization of the Euler-Poincarè equations:

$$\frac{d}{dt}\sigma = \delta\dot{\xi} = \delta\Big(-\frac{1}{2}\langle \xi : \xi \rangle_{\mathfrak{g}} \Big) = -\langle \xi : \delta\xi \rangle_{\mathfrak{g}} = -\langle \xi : \sigma \rangle_{\mathfrak{g}}.$$

□

Remark 4. From equation (3.4), the linearization of the Lagrangian equations (on the full space TG) along a geodesic $(g(t), \xi(t)) = g(t)^{-1}\dot{g}(t)$, has the following form

$$\frac{d}{dt}\begin{bmatrix} \eta \\ \sigma \end{bmatrix} = \begin{bmatrix} -\operatorname{ad}_\xi & I \\ 0 & -\operatorname{sym}_\xi \end{bmatrix} \begin{bmatrix} \eta \\ \sigma \end{bmatrix},$$

where $\operatorname{ad}_\xi(\zeta) = [\xi, \zeta]_{\mathfrak{g}}$ and $\operatorname{sym}_\xi(\zeta) = \langle \xi : \zeta \rangle_{\mathfrak{g}}$ for all $\zeta \in \mathfrak{g}$. It is interesting to underline that the stability of geodesics appears now to be related to the eigenvalues of operators ad_ξ and sym_ξ on the Lie algebra $\mathfrak{g}$. While the first operator is determined by the geometry of the group G, the second involves also the metric structure on $\mathfrak{g}$.

4. Controllability of Mechanical Systems: in general and the Lie group case

We start by reviewing the recent results obtained in [Lew95b] and we restrict ourselves to simple mechanical systems with no potential energy. The Lie group case is dealt with along the lines of the preprint [Lew95a].

We first state the general results obtained in [Lew95b]. Let $q_0 \in Q$ and let U be a neighborhood of q_0. Denote with $0_{q_0} \in T_{q_0}Q$ the zero tangent vector at q_0. We define

$$\begin{aligned}\mathcal{R}_Q^U(q_0, T) = \{q \in Q \mid \; & \text{there exists a solution } (c, u) \text{ of } (3.1) \\ & \text{such that } c'(0) = 0_{q_0},\, c(t) \in U \text{ for } t \in [0, T], \text{ and } c'(T) \in T_q Q\}\end{aligned}$$

and denote

$$\mathcal{R}_Q^U(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}_Q^U(q_0, t).$$

Notice that in the definition of $\mathcal{R}_Q^U(q_0, \leq T)$ we restrict our interest to the zero section of TQ, that is the set of zero tangent vectors. We now introduce our notions of controllability.

Definition 7. We shall say that (3.1) is *locally configuration accessible* at $q_0 \in Q$ if there exists $T > 0$ such that $\mathcal{R}_Q^U(q_0, \leq t)$ contains a non-empty open set of Q for all neighborhoods U of q_0 and all $0 < t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *locally configuration accessible.*

We say that (3.1) is *small-time locally configuration controllable* (STLCC) at q_0 if it is locally configuration accessible at q_0 and if there exists $T > 0$ such that q_0 is in the interior of $\mathcal{R}_Q^U(q_0, \leq t)$ for every neighborhood U of q_0 and $0 < t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *small-time locally configuration controllable.*

We shall say that (3.1) is *equilibrium controllable* if, for q_1, q_2 equilibrium points of L, there exists a solution (c, u) of (3.1) where $c : [0, T] \to Q$ is such that $c(0) = q_1$, $c(T) = q_2$ and both $c'(0)$ and $c'(T)$ are zero.

□

We now need to recall the definition of the symmetric product. If X and Y are vector fields on Q, we define

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

to be the *symmetric product* of X and Y. If $\mathcal{V}$ is a family of vector fields on Q, we shall denote by $\overline{\text{Lie}}(\mathcal{V})$ the *involutive closure* of $\mathcal{V}$, i.e. the set of vector fields on Q defined by taking iterated Lie brackets of vector fields in $\mathcal{V}$. In like fashion we define $\overline{\text{Sym}}(\mathcal{V})$ to be the collection of vector fields obtained by taking iterated symmetric products of vector fields from $\mathcal{V}$ and we call this collection the *symmetric closure*. In the following, let $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$, where the Y_i are the input vector fields of the simple mechanical control system in equation (3.1). We say that a symmetric product from $\overline{\text{Sym}}(\mathcal{Y})$

is *bad* if it contains an even number of each of the vector fields in $\mathcal{Y}$. A symmetric product which is not bad is called *good.*

The main result in [LM95] can be stated as follows:

Theorem 5 (Lewis-Murray). *The system* (3.1) *is*

i) locally configuration accessible at $q_0 \in Q$ *if* $\operatorname{rank}(\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y}))(q_0)) = \dim(Q)$,

ii) STLCC at $q \in Q$ *if it is locally configuration accessible at* q *and if every bad symmetric product can be written as a linear combination of good symmetric products of lower order at* q*, and*

iii) equilibrium controllable if it is STLCC at each $q \in Q$.

□

We now focus our attention on group invariant mechanical systems. Recall the forced Euler-Poincaré equations as

$$\begin{aligned} \dot{g} &= g\,\xi \\ \dot{\xi} &= (\mathrm{ad}_\xi^* \xi^\flat)^\sharp + u^a y_a \end{aligned} \tag{4.1}$$

and the definition of the symmetric product on $\mathfrak{g}$ from Remark 1 as

$$\begin{aligned} \langle\cdot:\cdot\rangle_{\mathfrak{g}} &: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g} \\ (x,y) &\mapsto \langle x:y\rangle_{\mathfrak{g}} = -\big(\mathrm{ad}_x^* y^\flat + \mathrm{ad}_y^* x^\flat\big)^\sharp. \end{aligned} \tag{4.2}$$

Let $\mathcal{Y} = \{y_1, \dots, y_m\}$ be the set of the input vectors. We can state a slightly stronger version of the previous theorem:

Theorem 6. *The system* (4.1) *is*

i) locally configuration accessible if $\operatorname{rank}(\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y}))) = \dim(G)$ *and*

ii) equilibrium controllable if it is locally configuration accessible and if every bad symmetric product can be written as a linear combination of good symmetric products of lower order.

Proof. Pull all the computations back to the identity $\mathfrak{g}$. The independence on the base point $g \in G$ makes the local properties hold globally. □

Remark 5 (Invariance implies algebraic computation scheme). The controllability properties stated in the theorem are independent of the base point $g \in G$. as the invariance of the original system suggested. As a consequence, the conditions for configuration controllability of the original nonlinear system are now expressed in a purely algebraic way (no differentiation is required).

Remark 6 (Coordinate expressions for the symmetric product). Motivated by the meaning that the symmetric product (4.2) now assumes. we look for matrix and local coordinate expressions for it.

Let the metric tensor $g = \mathbb{I}$ (moment of inertia) and recall the definition of the input vectors $y_a = (f^a)^\sharp = \mathbb{I}^{-1} f^a$. Then we can write matrix expressions for the Euler-Poincaré equations

$$\begin{aligned}\mathbb{I}\,\dot{\xi} &= \mathrm{ad}_\xi^T(\mathbb{I}\,\xi) + u^a\,\mathbb{I}y_a \\ &= \mathrm{ad}_\xi^T(\mathbb{I}\,\xi) + u_a f^a,\end{aligned}$$

(where we redefine $u_a = u^a$) and for the symmetric product

$$\begin{aligned}\langle x : y\rangle_{\mathfrak{g}} &= -(\mathrm{ad}_x^* y^\flat + \mathrm{ad}_y^* x^\flat)^\sharp \\ &= -\mathbb{I}^{-1}\big(\mathrm{ad}_x^T\,\mathbb{I}y + \mathrm{ad}_y^T\,\mathbb{I}x\big). \qquad (4.3)\end{aligned}$$

Additionally, let $\{e_1, \ldots e_n\}$ be a basis of the the Lie algebra $\mathfrak{g}$. We can define the structure coefficients c_{ij}^k and the symmetric coefficients γ_{ij}^k by

$$c_{ij}^k = [e_i, e_j]^k \quad \text{and} \quad \gamma_{ij}^k = \langle e_i : e_j\rangle^k,$$

where we compute

$$\gamma_{ij}^k = -\mathbb{I}^{hk}(\mathbb{I}_{il}c_{jh}^l + \mathbb{I}_{jl}c_{ih}^l).$$

Remark 7 (On the symmetric product). On the linear space $\mathfrak{g}$, we have a Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}}$ and a metric structure $\mathbb{I}$. The symmetric product is indeed a combination (maybe the simplest [2]) of the two and, as it is suggested by for example equation (4.3), measures whether the two structures "commute" one with each other.

Indeed, recall from the discussion in Subsection 2.1 that if the metric is bi-invariant the symmetric product vanishes. That is, if the adjoint action of the group G is an orthogonal representation on the inner product space $(\mathfrak{g}, \mathbb{I})$, then the symmetric product vanishes.

Remark 8 (The symmetric product on $\mathfrak{g}^*$). Given the strong interplay between $\mathfrak{g}$ and $\mathfrak{g}^*$ in the definition of the symmetric product, let us define the dual operation on $\mathfrak{g}^*$:

$$\begin{aligned}\langle\cdot:\cdot\rangle_{\mathfrak{g}^*} &: \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^* \\ &(\alpha, \beta) \mapsto \langle\alpha : \beta\rangle_{\mathfrak{g}^*} = \left\langle\alpha^\sharp : \beta^\sharp\right\rangle_{\mathfrak{g}}^\flat. \qquad (4.4)\end{aligned}$$

Then one can easily compute

$$\langle\alpha : \beta\rangle_{\mathfrak{g}^*} = -\,\mathrm{ad}_{\alpha^\sharp}^*\,\beta - \mathrm{ad}_{\beta^\sharp}^*\,\alpha$$

Let us now consider the coadjoint action of G on $\mathfrak{g}^*$. Recall that the infinitesimal generator of this action corresponding to an element $\xi \in \mathfrak{g}$ is given by $\xi_{\mathfrak{g}^*} = -\,\mathrm{ad}_\xi^*$, see [AM87, Section 4.1]. Then we can write

$$\langle\alpha : \beta\rangle_{\mathfrak{g}^*} = (\alpha^\sharp)_{\mathfrak{g}^*}(\beta) + (\beta^\sharp)_{\mathfrak{g}^*}(\alpha).$$

[2] If one were to guess it, this is actually the simplest possible definition.

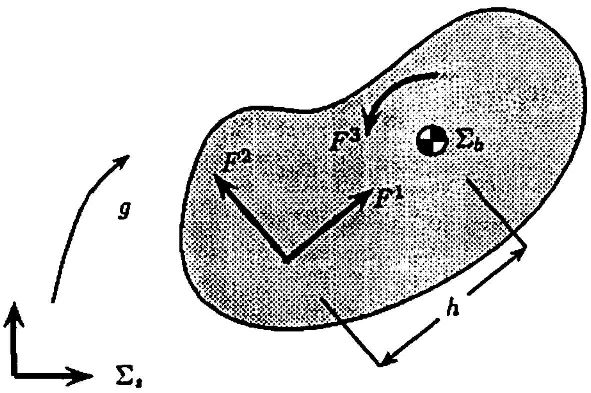

FIGURE 1. Forced planar rigid body: the configuration variable $g \in SE(2)$ determines the position of the body frame Σ_b with respect to the spatial frame Σ_s. Notice the positions of application of the various forces.

5. EXAMPLES

We illustrate the results through two examples: the forced planar rigid body and the rigid body with external torques.

Example 1 (Forced planar rigid body). The mechanical system is depicted in Figure 1. Let $g \in SE(2)$ be the configuration of the system. Let (x, y, θ) be the standard coordinate chart for $SE(2)$ and let

$$\begin{aligned}\widetilde{e}_1 &= \cos\theta\frac{\partial}{\partial x} - \sin\theta\frac{\partial}{\partial y}\\ \widetilde{e}_2 &= \sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\\ \widetilde{e}_3 &= \frac{\partial}{\partial\theta}\end{aligned} \tag{5.1}$$

be a left-invariant basis [3] for $T_gSE(2)$ and let $\{\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^3\}$ be its dual basis. Denote with $\langle\cdot,\cdot\rangle$ the pairing on the bundle $TSE(2)$ and its value at the identity $\mathfrak{se}(2)$. Then we write $\langle\widetilde{e}^i, \widetilde{e}_j\rangle = \langle e^i, e_j\rangle = \delta^i_j$, where $\{e_i\}$ and $\{e^i\}$ are the corresponding bases of $\mathfrak{se}(2)$ and $\mathfrak{se}(2)^*$. Consider the left-invariant metric tensor

$$\begin{aligned}\mathbb{I} &= m dx \otimes dx + m dy \otimes dy + J d\theta \otimes d\theta\\ &= m(\widetilde{e}^1 \otimes \widetilde{e}^1 + \widetilde{e}^2 \otimes \widetilde{e}^2) + J\widetilde{e}^3 \otimes \widetilde{e}^3.\end{aligned}$$

The control inputs consist of forces applied at a distance $h \neq 0$ along the x body-axis and a torque about the center of mass, see Figure 1. We write

[3] This choice corresponds to the natural left-invariant basis $\Omega = g^{-1}dg$ of $\mathfrak{g}^*$. See [SW86, Chapter 7].

the control one-forms (in $T_g^*SE(2)$) as

$$F^1 = \tilde{e}^1, \qquad F^2 = \tilde{e}^2 - h\tilde{e}^3, \qquad F^3 = \tilde{e}^3$$

from which we compute the input vectors (in $\mathfrak{se}(2)$) as

$$y_1 = \frac{1}{m}e_1, \qquad y_2 = \frac{1}{m}e_2 - \frac{h}{J}e_3, \qquad y_3 = \frac{1}{J}e_3.$$

For sake of completeness we report the forced Euler-Poincaré equations in matrix notation as

$$\begin{bmatrix} m\dot{v}_x \\ m\dot{v}_y \\ J\dot{\omega} \end{bmatrix} = 2\omega \begin{bmatrix} -v_y \\ v_x \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u^1 + \begin{bmatrix} 0 \\ 1 \\ -h \end{bmatrix} u^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u^3,$$

where $\xi = v_x e_1 + v_y e_2 + \omega e_3$ is the body-fixed body velocity.

We can now rederive every Lie bracket and symmetric product computed in [LM95]. We consider all the possible combinations of inputs:

Two forces y_1, y_2: The system is equilibrium controllable, since

$$\langle y_1 : y_2 \rangle_{\mathfrak{se}(2)} = -\frac{h}{Jm}e_2,$$

(hence local configuration accessibility follows) and the only relevant "bad" product $\langle y_2 : y_2 \rangle_{\mathfrak{se}(2)} = (2h/J)y_1$.

Force y_1 and torque y_3: The system is equilibrium controllable, since

$$\langle y_1 : y_3 \rangle_{\mathfrak{se}(2)} = \frac{1}{Jm}e_2$$

and all the "bad" symmetric products vanish.

Force y_2 and torque y_3: The system is locally configuration accessible, since

$$\langle y_2 : y_3 \rangle_{\mathfrak{se}(2)} = -\frac{1}{Jm}e_1$$

but the "bad" product $\langle y_2 : y_2 \rangle_{\mathfrak{se}(2)}$ is independent from $\mathrm{span}\{y_2, y_3\}$ (i.e. the sufficient condition fails).

One input cases: If only the force y_1 or the torque y_3 is available, all the symmetric products and Lie brackets vanish, so that the system is not locally configuration accessible.

If the only input we have is the force y_2, then $\mathrm{span}\{y_2, \langle y_2 : y_2 \rangle_{\mathfrak{se}(2)}, [y_2, \langle y_2 : y_2 \rangle_{\mathfrak{se}(2)}]_{\mathfrak{se}(2)}\} = \mathfrak{se}(2)$, implying local configuration accessibility (however the presence of a "bad" product makes the sufficient test for controllability fail).

The controllability results for the forced planar rigib body are summarized in the following table: the symbol "■" means failure of the rank test.

Control inputs	Local configuration accessibility	Equilibrium controllability
Force y_1 and force y_2	yes	yes
Force y_1 and torque y_3	yes	yes
Force y_2 and torque y_3	yes	? (sufficient test fails)
Force y_1	■	■ (of course)
Force y_2	yes	? (sufficient test fails)
Torque y_3	■	■ (of course)

□

Example 2 (Rigid body with external torques). Let $g \in SO(3)$ be the configuration of the system (that is the attitude of the rigid body). Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathfrak{so}(3)$, such that $[e_i, e_j]^k = \epsilon_{ij}^k$ is the alternating tensor. Also, let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be the corresponding left-invariant basis for $T_gSO(3)$. Consider the metric tensor

$$\mathbb{I} = J_i \tilde{e}^i \otimes \tilde{e}^i,$$

where summation over i is implied. The control inputs consist of external torques about the center of mass. We write the control one-forms (in $T_g^*SO(3)$) as $F^i = \tilde{e}^i$ (hence the inputs are exerted along the principal inertia axes), and we compute the input vectors (in $\mathfrak{so}(3)$) as $y_i = e_i/J_i$.

For sake of completeness we report the forced Euler-Poincaré equations in matrix notation as

$$\mathbb{I}\dot{\xi} = \mathbb{I}\xi \times \xi + u^a e_a,$$

where we denote with $\mathbb{I}$ both the metric tensor and its matrix representation. Note that ξ is the body-fixed angular velocity, sometimes denoted with ω.

We can now derive tests for local configuration accessibility and equilibrium controllability. Note that our results show a close similarity to the standard treatment in [NvdS90].

Two actuators: Assuming the two actuators are aligned along the first two principal axes,

$$\langle y_1 : y_2 \rangle_{\mathfrak{so}(3)} = \frac{J_2 - J_1}{J_1 J_2 J_3} e_3,$$

and local configuration accessibility follows for $J_1 \neq J_2$ (asymmetric rigid body). Also, since $\langle y_i : y_i \rangle_{\mathfrak{so}(3)} = 0$ for $i = 1, 2$, all "bad" product vanish and the system is equilibrium controllable. (Note that condition (3.53) in [NvdS90] for strong accessibility of the "velocity subsystem" *surprisingly* agrees with our condition for configuration accessibility.)

For the symmetric rigid body, that is when $J_1 = J_2$, the system is locally configuration controllable through $[y_1, y_2]_{\mathfrak{so}(3)} = e_3$.

One actuator: Assuming that the available actuator is aligned with any of the principal axes, we have

$$\langle y_1 : y_1 \rangle_{\mathfrak{so}(3)} = 0 \quad \text{and} \quad [y_1, y_1]_{\mathfrak{so}(3)} = 0,$$

so that the system is not locally configuration accessibile. For y_1 not aligned along any of the principal axes, local configuration accessibility is "generally" achieved, but the sufficient conditions for STLCC fails.

□

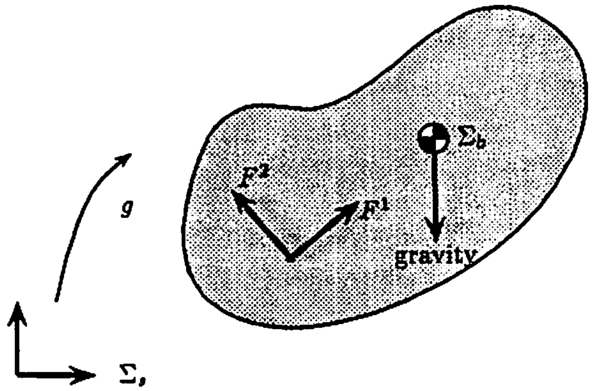

FIGURE 2. Forced planar rigid body with gravity. The full group symmetry is broken, but still a lot of structure is present.

6. COMMENTS

In this report controllability problems for mechanical systems on Lie groups have been analyzed using invariant affine connections. For these systems, the general theory developed in [Lew95b] leads to a simple algebraic test and to a detailed treatment of important examples, like the rigid body with external torques and the forced planar rigid body.

While performing this analysis, we have also understood how to perform Lagrangian reduction within a Riemannian geometry context. In particular, we showed how to derive the Euler-Poincarè equations (the reduced Lagrange's equations) and the reduced Jacobi equation. More generally, we have shown the importance of the Lie group case as a testbed for new ideas and techniques in the area of "nonlinear control of mechanical systems."

Future directions of research will focus on the following themes:

1. Theoretical controllability issues on Lie groups deserve some dedicated attention. In the examples, it appears that the test for equilibrium controllability also guarantees strong accessibility of the "velocity subsystem". This might lead to instructive generalizations of the work in [Bon84] and [Bai81].
2. In the literature, the use of affine connections for control of mechanical systems is not as wide as one could expect. In particular, stability problems are usually addressed within the Hamiltonian framework, see for example the Energy-Momentum-Casimir method. From a Lagrangian viewpoint, it is instructive to look at affine connections and (for example) sectional curvature as tools to design feedback controls laws and assess stability.
3. Also, we plan on looking at the constructive controllability problem. In particular, our goal is a design methodology for the stabilization of

(equilibrium controllable) invariant systems with time-varying inputs, see for example [Leo95].

4. Eventually we would like our approach to encompass systems with symmetry breaking effects. In particular, we refer to the toy-example depicted in Figure 2. The forced planar rigid body with gravity constitutes a rather simple example, but with all of the most interesting features (and it is already of interest!). The theory of semidirect products might be instrumental in the understanding of the problem.

Acknowledgements

This work is motivated by recent advances in the controllability theory for mechanical systems [LM95] and is the fruit of a series of interesting discussions with Andrew Lewis.

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