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Introduction

Needless to say, the study of fluid and plasma systems is - except in the ideal classroom setting - a quite formidable undertaking. It is my hope that through some of the techniques I've seen during this course, I can obtain greater insights into some problems concerning the stability of coronal loops.

Not entirely unrelated to the study of stability in plasma systems, is the availability of Hamiltonian - like structures which reflect the time evolution in terms of conserved quantities. With a view toward a deeper understanding of the place Lie-Poisson structures occupy in this regard, I've been studying the representation of the Euler equations for ideal fluid flow as Lie-Poisson equations for a "Hamiltonian" in terms of vorticity.

Ideal Fluid Flow as a Lie-Poisson System

For simplicity, assume fluid flow takes place in an open, simply connected region W of $\mathbb{R}^2$. Recall that Euler's equations for ideal fluid flow can be written as:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla} p$$

$$\operatorname{div}(\vec{v}) = 0$$

$$\vec{v}(\vec{x}, 0) = \vec{v}_0(\vec{x})$$

Here $\vec{x} = (x, y) \in \mathbb{R}^2$, $\vec{v}_0$ is a given divergence free vector field on W and $p : W \to \mathbb{R}$ subject to the conditions

$$\operatorname{div}(\vec{v}\cdot\vec{\nabla}\vec{v} - \vec{\nabla} p) = 0$$

$$(\vec{v}\cdot\vec{\nabla})\vec{v} - \vec{\nabla} p \parallel \partial W .$$

Begin by considering as the configuration space of a Hamiltonian system, the collection of area preserving diffeomorphisms, $\mathcal{D}(W)$, mapping W to itself. $\mathcal{D}(W)$ is a group under composition.

Indeed, $\mathcal{D}(W)$ is dealt with as if it were a Lie group, but I have no personal first hand knowledge of this fact. Proceeding as though this assumption is sound, one considers $T^* \mathcal{D}(W)$ as the phase space. The Lie algebra of $\mathcal{D}(W)$, $\mathfrak{X}(W)$ is then identified with the vector fields $\vec{v}$ on W satisfying

$$\operatorname{div} \vec{v} = 0 , \quad \vec{v} \parallel \partial W . \quad \text{If} \quad \xi \in \mathfrak{X}^*(W)$$

$$\xi(v) = \int_W v \cdot \varphi_\xi \, dA$$ where φ_ξ is an L^2 representative of the functional ξ, and "$\cdot$" indicates the standard dot product. {from here on we suppress the little "$\rightarrow$" sign except when necessitated for clarity}.

If $\varphi_\xi = df$ where $f : W \rightarrow \mathbb{R}$ is C^1, then

$$\operatorname{div}(fv) = (\operatorname{grad} f)\cdot v + f \operatorname{div} v$$

$$= df \cdot v \qquad \text{since } \operatorname{div} v = 0 .$$

Hence, letting n be a unit vector normal to ∂W, and recalling that $v \parallel \partial W$, Gauss' theorem gives

$$0 = \int_{\partial W} (fv \cdot n)\, ds = \int_W \operatorname{div}(fv)\, dA = \xi(v) .$$

It follows that every exact one form annihilates $\mathfrak{X}(\mathcal{W})$, and thus one identifies $\mathfrak{X}^*(\mathcal{W})$ with the one forms on $\mathcal{W}$ modulo exact one forms, written $\Omega^1(\mathcal{W}) / d\Omega^0(\mathcal{W})$.

From the Hodge decomposition theorem for manifolds with boundary, given $\xi \in \Omega^1$ there is a function ϕ and a one-form η such that $\xi = d\phi + \eta$ where η is tangent to $\partial\mathcal{W}$, $\delta\eta = 0$ and $d\phi$ and η are L^2 orthogonal. Thus, any element in $\Omega^1 / d\Omega^0$ is given solely by η.

One interprets the two form $\omega = d\xi$ as the vorticity. The condition $\delta\xi = 0$ implies $\operatorname{div} \xi = 0$, thus $\xi \in \mathfrak{X}(\mathcal{W})$. For a given ω write $\xi(\omega)$ = the 1-form giving rise to ω as above. Then define on $\mathfrak{X}^*(\mathcal{W})$

$$H(\omega) = \frac{1}{2} \int \| \xi(\omega) \|^2 \, dA .$$

H defines a right-invariant function on $\mathfrak{X}^*(\mathcal{W})$.

This follows from the fact that the energy, given in terms of the velocity, is right invariant on $T\mathcal{D}(\mathcal{W})$.

Euler's equations in vorticity form read

$$\frac{D\omega}{Dt} - \omega \cdot \nabla v = 0 \quad , \quad \omega = \nabla \times v$$
$$\operatorname{div} v = 0 \quad , \quad v \parallel \partial \mathcal{W}$$

One verifies either directly or by reduction that these Euler's equations are equivalent to the Lie Poisson equations on $\mathfrak{X}^*(\mathcal{W})$ given by $\dot{F} = \{F, H\}$ where one puts
$$\{F, G\}(\omega) = \int_{\mathcal{W}} \left\langle \omega, \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right] \right\rangle dA .$$

Here $\frac{\delta F}{\delta \omega}$ identifies the element in $\mathfrak{X}(\mathcal{W})$ for $F : \mathfrak{X}^*(\mathcal{W}) \to \mathbb{R}$, which induces the linear functional $DF(\omega) : (\Omega^1 / \Omega^0) \to \mathbb{R}$.

A direct computation yields that $\dot{F} = \{F, H\}$ is equivalent to $\frac{\partial \omega}{\partial t} + \mathcal{L}_v \omega = 0$ – the

Lie form for the vorticity equations, which implies that ω is Lie-transported by the flow.

Coadjoint Actions and Orbits

The action of $\mathcal{D}(\Omega)$ on $\mathfrak{X}^*(\Omega)$ is given from the right, thus a modification of the result on p. 261 An introduction to mechanics and symmetry — shows that this action is essentially $(\eta, \omega) \mapsto \eta^* \omega$.

The coadjoint orbits are then the sets

$$\mathcal{O}_\omega = \{ \eta^* \omega \mid \eta \in \mathcal{D}(\Omega) \},$$

and from this it follows that a given vorticity ω with $\omega(0) = \omega_0$, is constrained to $\mathcal{O}_{\omega_0}$.

Also, the symplectic structure on $T_\omega \mathcal{O}_\omega$ is given by $\Omega_\omega(\pounds_{v_1}\omega, \pounds_{v_2}\omega) = \int \omega(v_1, v_2)\, dA$.

Indeed, by the the general formula for symplectic structures on coadjoint orbits, and by "right" invariance, one obtains

$$\Omega_\omega(\mathcal{L}_{v_1}\omega, \mathcal{L}_{v_2}\omega) = \langle \omega, -[v_1, v_2] \rangle$$

for $\omega = d\xi$, the above gives

$$-\int \xi \cdot [v_1, v_2]\, dA$$

$$= -\int \xi \cdot \mathcal{L}_{v_1} v_2 \, dA = \int (\mathcal{L}_{v_1}\xi)\, v_2 \, dA - \int \mathcal{L}_{v_1}(\xi \cdot v_2)\, dA$$

$$= \int \left(i_{v_1} d\xi + d i_{v_1}\xi\right) \cdot v_2 \, dA - \int \operatorname{div}(v_1)\, (\xi \cdot v_2)\, dA$$

$$= \int (i_{v_1} d\xi)\, v_2 \, dA \quad \text{since} \quad \operatorname{div} v_1 = \operatorname{div} v_2 = 0 .$$

But $(i_{v_1}\omega)\, v_2 = \omega(v_1, v_2)$ and the result follows.

<u>Clebsch Variables and Symplectic Structures</u>

<u>Clebsch variables</u> for a Poisson manifold P, are defined to be a Poisson map Ψ from a symplectic manifold Q to P. The <u>gauge</u> <u>transformations</u> associated to Ψ are symplectic transformations φ such that

$$\Psi \circ \varphi = \Psi .$$

The Clebsch variables for this flow turn out to be momentum maps on the action of $\mathcal{D}(W)$ on Q. By letting $\mathcal{D}(W)$ act on the space $\mathcal{F}$ = real valued L^2 functions on W via $\mathcal{D}(W) \times \mathcal{F} \longrightarrow \mathcal{F}$ by

$$(\eta, f) \longmapsto f \circ \eta^{-1}$$

one obtains a symplectic left action on $T^*\mathcal{F}$, and the corresponding momentum map is $(\eta, f) \longmapsto d\eta \wedge df$.

The classical Clebsch representation is obtained by considering the case where $\omega = d\eta \wedge df$ is exact. If this is the case, one writes $\nu(\omega) = d\xi + \eta\, df$.

Another approach is to use the two dimensional symplectic structure on $\mathbb{R}^2$, ie. where $\{F, G\} = \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial y}$. In this scheme, one maps $(\lambda, \mu) \rightarrow \{\lambda, \mu\} = \omega$, which turns out to be a Poisson map.

Summary

There is still much I'd like to learn. There are interesting applications of Clebsch variables and the symplectic structures on $\mathcal{O}_\omega$ in the study of point vortices and vortex patches. I'll certainly be spending some time deepening my understanding of these ideas this summer, in addition to reviewing the paper by Arnold {I've still not gotten to it}.

References

1) Marsden and Ratiu; An Introduction to Mechanics and Symmetry

2) Marsden and Weinstein; Coadjoint Orbits, vortices and Clebsch Variables for incompressible fluids