PATH PREDICTION FOR AN EARTH-BASED DEMONSTRATION BALLOON FLIGHT

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1. Introduction

1.1. **Motivation.** The Cassini-Huygens spacecraft mission, sent to study Saturn and its moons, revealed some incredible observational data regarding one of Saturn's moons, Titan. The data sent back to Earth demonstrates the likely existence of high-latitude hydrocarbon oceans and equatorial sand dunes. The hydrocarbon oceans are currently the only open bodies of liquid found outside of our planet [4]. The rocks on the surface are likely made up of water-ice pebbles, analogous to the silicate sands found on Earth [3]. While life is unlikely to exist on Titan, needless to say, its uncanny similarities to Earth have sparked the interest and curiosity of the scientific community [5].

NASA intends on going back to Titan, likely using lighter-than-air vehicles, such as hot-air balloons, that would be able to operate and gather data on Titan for several years [6]. Ideally, the balloons would only need to use vertical controls to navigate to high-priority science targets. However, if necessary, the balloons could also be equipped with horizontal controls. Using this and the winds on Titan, it would be hoped that the balloons could navigate between different regions of Saturn's moon [1].

1.2. **The Problem.** While the balloon is on Titan, intervening with the flight-plan on a short time-scale is not possible. Therefore, it is necessary for the balloon to make decisions autonomously and still be able to reach particular regions of interest on Titan in order to gather quality data. Ideally, the balloon would be able to exploit the wind fields of Titan as much as possible to minimize the power and control requirements.

It is not exactly clear what type of information, particularly regarding wind fields, that the balloon will have. A global wind model for the atmosphere of Titan exists though it is not very robust. Moreover, this model can help the balloon navigate Titan on a global scale, but locally may not be of much use. Even with perfect information this problem is not easy to solve.

It is of note that the current target date for launch may not be until 2016. Until then, some experiments will be run on Earth to test methods that can be translated to solve the problem on Titan. If an approach is to succeed on Titan, it would be hoped that it succeeds on Earth, whose atmospheric models are much more robust and reliable due to a greater availability of accurate data and verifiable results. Moreover, it is much easier to test the results on Earth as computed data can be relatively easily compared to experimental results. Obviously, the hope is that the computed results would closely match the reality. If so, the techniques used to solve the problem on Earth can be translated to approach a similar problem on Titan.

The atmospheric model we will be using for Earth is called the Weather Research and Forecasting (WRF) Model, which produces the wind fields for a specific region at a given time [2]. I will continue to learn more about this model, its capabilities, and its shortcomings.

The output of the WRF Model can then be used as the input into a software package called Newman which computes particle trajectories and Lagrangian Coherent Structures (LCS) using finite-time Lyapunov exponents [8]. This allows us to describe the likely path taken by a passive balloon on a global scale, and, more importantly, shed light into the necessary power requirements for the balloon to reach other regions of interest (with possibility of having to overcome transport barriers).

On a local scale, a tool called Discrete Mechanics and Optimal Control is able to determine optimal trajectories based on a given wind field. It has been demonstrated for a simple underwater glider in a two-dimensional time-independent water flow. Being able to extend this for three dimensions with variable wind velocities is essential.

As mentioned, however, it is not entirely clear what information will be available to the balloon in order to calculate these optimal (i.e. minimal control) trajectories. If one has perfect information about current wind velocities, one approach is to divide the trajectory between the two points into many intermediate target steps and compute the optimal trajectory from one intermediate step to the next. At the very least, this can deal with the issue of variable winds (as long as the time steps are shortened sufficiently), however its effectiveness given realistic information about the winds needs to be investigated. One assumption that may slightly simplify the problem is that highly-variable winds are not expected near the surface of Titan. On the other hand, a technique operating under this assumption must be studied in case the unexpected happens. Hopefully, over the next few years, more information about the winds on Titan will be revealed and the forecasting model will become more robust.

1.3. **Related Ongoing Work.** Much of this project could tie in with another one investigating transport barriers in Titan's atmosphere via Titan's wind model as input to calculate LCS. Along these lines, this project would also investigate optimal strategies to travel between regions of Titan though likely on a more global scale than the Earth-based project. Hopefully, many of my results could help this project and vice versa.

A similar problem that requires the use of many of the same tools (i.e. LCS) is optimization (i.e. minimization of fuel expenditure) of the spacecraft's trajectory to Titan. It is possible that such trajectories could include a flyby of Enceladus (another moon of Saturn) and other moons of scientific interest.

There are other issues that need to be dealt with for the Titan mission. The exact type of balloon that will be used is still being decided. Also, ideally, the balloon would not have to land making it necessary for a robotic arm or some other machine, like a rover, to do the surface sampling.

The conglomeration of all these projects (and this is far from an inclusive list) will help enable the Titan mission to achieve the goals of advancing our understanding of extraterrestrial planets and moons.

2. Preparation

In order to understand the tools being used to solve the problem, I have a steep learning curve to overcome. As a result, I have spent some time studying ODE solvers and will study differential forms and tensor analysis. Discrete Mechanics and Optimal Control (DMOC) [7] describes a method to calculate an optimal trajectory subject to some physical constraints. It is mentioned that this method does not require deriving the Euler-Lagrange equations of motion for the system, unlike the other methods that require ODE solvers (i.e. shooting methods). To understand the context in which this method was developed, I studied numerical methods to solve ordinary differential equations. Two specific numerical methods mentioned, the Euler-based approach and the midpoint rule will discussed in greater depth here.

2.1. ODE Solvers.

There are many numerical methods to solve ordinary differential equations with each having its own advantages and disadvantages. Thus it is important to appropriately choose an ODE solver that fits a given problem.

ODE solvers can be split into two categories, explicit and implicit. Explicit solvers are generally less expensive than implicit solvers, however they can also be less stable. On the other hand, certain types of problems, for example stiff systems, require an implicit solver. An example of a stiff system is the Burridge-Knopoff model of earthquakes. Consider several block masses connected to each other via springs. Suppose we were to pull the end block slowly over time via a spring. Due to static friction, the blocks will not move until the accumulated force matches the static friction of the blocks (which could take years) in which case they move very suddenly. The differential equations that govern this model are considered stiff.

A comparison of explicit and implicit methods will be shown using Euler's method for a pendulum. A third semi-implicit solver, which is a symplectic one, will also be shown. I will use properties of differential forms in the derivation of the symplectic solver.

Let us consider the following ODE

$$y' = f(t, y).$$

with the initial condition

$$y(0) = y_0.$$

We can actually consider y to be a vector (i.e. a system of ODEs) and all the results still apply.

The explicit Euler method is

$$y_{n+1} = y_n + h \cdot f(t_n, y_n).$$

where h is the step size. If we consider an approximation of derivatives

$$\frac{y(t+h) - y(t)}{h} \approx y',$$

then it is clear where the explicit Euler method comes from. The Euler method is an example of multistep ODE solvers (in this case, actually, Euler is a single step method) which use previous values to determine the current one. An alternative type of ODE solvers are the so-called Runge-Kutta methods.

We can consider the explicit Euler method to be like a left-end point method for integration. From this perspective, the implicit Euler method

$$y_{n+1} = y_n + h \cdot f(t_{n+1}, y_{n+1})$$

is like a right-end point method for integration. If $f(t, y) = -ky$ and k was sufficiently large (i.e. stiff system) the explicit method will converge for only a sufficiently small step size. However, this creates numerical problems because such a small step size could cause the solution to lose accuracy and create errors in this solution. In this case, an implicit method is appropriate.

However, it is of note that y_{n+1} is on both sides of the equation for the implicit method forcing us to solve a nonlinear equation. One approach to this is to use fixed-point iteration to calculate y_{n+1}. Let us consider (note that in this case, y_n is a constant)

$$X = y_n + h \cdot f(t_{n+1}, X) = g(X).$$

The fixed point iteration

$$X^{(m+1)} = g(X^{(m)})$$

should eventually converges to X under certain conditions on the function g. In particular, if g is differentiable and $|g'(X)| < 1$ then this iteration is guarunteed to converge for a sufficiently close initial approximation. In the particular case of ODEs, this condition can be controlled by selecting a sufficiently small step size h. This iterative method allows us to compute y_{n+1}. However, fixed point interation is not appropriate for stiff systems.

I tested these methods for the equation of a pendulum

$$\theta'' + \sin(\theta) = 0.$$

where $t \in [0, 50]$. If we make the substitution $x = \theta$ and $y = \theta'$, then we obtain the system of differential equations

$$x' = y$$

$$y' = -\sin(x).$$

First, consider a step size of $h = 0.1$ with initial conditions $(x_0, y_0) = (0.1, 0)$. The explicit Euler method gives the phase diagram in Figure (4.1) included in Appendix (4). One immediately notices that this method does not conserve energy and in fact increases in energy. If we decrease the step size, this becomes less noticeable but is still a major limitation of the method.

The implicit Euler method for the same initial conditions gives the phase diagram in Figure (4.2). Again, one notices that this method does not conserve energy and in fact decreases in energy. Again, decreasing step size makes this less noticeable, but if it was essential to accurately track the location of the pendulum, these methods would not be ideal.

One can use Richardson extrapolation to improve accuracy. Sparing the details of the method, the idea is to take linear combinations of solutions obtained from relatively large step sizes to improve the number of digits of the solution. This allows us to use a low-order method, such as implicit Euler, with a relatively large step size to obtain accurate results. I will define the order of a method later.

Let us also derive a symplectic method which has advantages for Hamiltonian systems. We can derive a symplectic method based on the following change of variables

$$p' = p - h \cdot \sin(\frac{q' + q}{2})$$

$$q' = q + h \cdot \frac{p' + p}{2}$$

where p is impulse and q is position. Using basic properties of differential forms, let us compute $dp' \wedge dq'$. We have that

$$dp' = dp - h \cdot \cos(\frac{q' + q}{2})(\frac{dq' + dq}{2})$$

$$dq' = dq + h \cdot \frac{dp' + dp}{2}.$$

We rewrite these equations as

$$dp' + \frac{h}{2} \cdot \cos(\frac{q' + q}{2})dq' = dp - \frac{h}{2} \cdot \cos(\frac{q' + q}{2})dq$$

$$dq' - \frac{h}{2} \cdot dp' = dq + \frac{h}{2}dp.$$

On the left hand side, we have

$$\begin{aligned}(dp' + \frac{h}{2} \cdot \cos(\frac{q' + q}{2})dq') \wedge (dq' - \frac{h}{2} \cdot dp') &= dp' \wedge dq' - \frac{h}{4} \cdot \cos(\frac{q' + q}{2})dq' \wedge dp'. \\ &= (1 + \frac{h}{4} \cdot \cos(\frac{q' + q}{2}))dp' \wedge dq'\end{aligned}$$

and on the right hand side, we have

$$\begin{aligned}(dp - \frac{h}{2} \cdot \cos(\frac{q' + q}{2})dq) \wedge (dq + \frac{h}{2}dp) &= dp \wedge dq - \frac{h}{4} \cdot \cos(\frac{q' + q}{2})dq \wedge dp \\ &= (1 + \frac{h}{4} \cdot \cos(\frac{q' + q}{2}))dp \wedge dq.\end{aligned}$$

Therefore, we have that

$$dp' \wedge dq' = dp \wedge dq.$$

This demonstrates that the iterative method

$$x_{n+1} = x_n + h \cdot (\frac{y_{n+1} + y_n}{2})$$

$$y_{n+1} = y_n - h \cdot \sin(\frac{x_{n+1} + x_n}{2})$$

conserves area in the configuration space. In fact, this can be also shown for any time-independent Hamiltonian system. One may view this as the exact solution of the perturbed Hamiltonian system. For this case, the Hamiltonian is

$$H(x, y) = \frac{y^2}{2} - \cos(x).$$

A symplectic method exists for the Euler-based approach as well. The one derived above, written more generally as

$$z_{n+1} = z_n + h \cdot f(\frac{z_{n+1} + z_n}{2})$$

is known as the implicit midpoint rule.

A comparison of all three methods with the same initial condition is shown in Figure (4.3). This clearly shows the advantages of the symplectic (implicit midpoint) method. Using this method, the phase diagram for the pendulum is shown in Figure (4.4).

I will use the Euler methods and the midpoint rule to motivate definitions of stability properties for different ODE solvers. Let us consider the so-called test problem

$$y' = \lambda y$$

where λ may be complex. The exact solution of the test problem is

$$y(t) = y_0 \exp(\lambda t).$$

We know that this solution is bounded for all $t > 0$ as long as $\Re e(\lambda) < 0$.

Let us first consider the explicit Euler method. We have that

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda)^n y_0.$$

In order for this to be bounded we require that

$$|1 + h\lambda| \leq 1.$$

The region of $h\lambda$ in which the solution is bounded is a circle of radius 1 centered at $(-1, 0)$ in the complex plane.

In general, the region of absolute stability is the set of values for $h\lambda$ for which the solution of the test problem will remain bounded as $n \to \infty$. In this case, the region of absolute stability does not match the properties of the test problem. A method is said to be stable if the region of absolute stability includes the origin.

The implicit Euler method is considered to be over-stable since the region of absolute stability is

$$|1 - h\lambda| \geq 1$$

which is the entire complex plane except for a circle of radius 1 centered at $(1, 0)$ in the complex plane.

Now we consider the implicit midpoint rule. We have

$$z_{n+1} = z_n + h \cdot \lambda \frac{z_{n+1} + z_n}{2}$$

which gives us

$$z_{n+1} = \left(\frac{1 + \frac{h \cdot \lambda}{2}}{1 - \frac{h \cdot \lambda}{2}}\right)^n z_0.$$

This solution is bounded when

$$\left|\frac{1 + \frac{h \cdot \lambda}{2}}{1 - \frac{h \cdot \lambda}{2}}\right| \leq 1$$

or, more importantly, when $\Re e(h\lambda) < 0$. This means that the midpoint method is A-stable.

Let us now consider the notion of the order of ODE solvers. The order, p, of the ODE solver describes local behavior of the scheme. Specifically, as we advance to the next iteration of the solver (i.e y_n to y_{n+1}), the local error is of $\mathcal{O}(h^{p+1})$. If $p \geq 1$ (for multistep methods) and the method is stable then the method is convergent (the converse is also true).

The Euler methods are first order schemes while the implicit midpoint method is second order. The classic Runge-Kutta method is fourth order. In general, higher order methods are preferred so that computations can be performed with a relatively large step size.

While I will continue studying numerical methods of solving ordinary differential equations, I will also study differential forms and tensor analysis. These subjects play a large role in many subject areas including Hamiltonian mechanics and dynamical systems. I have been and will continue reading books such as *Differential Forms with Applications to the Physical Sciences* by Harley Flanders and *Tensor Analysis: Theory and Applications to Geometry and Mechanics of Continua* by I.S. Sokolnikoff.

3. Future Work

3.1. **Weather Research and Forecasting Model.** I am starting to learn to run the WRF model. The model is a very powerful tool that will allow me to produce wind fields for any place and time. It is a numerical weather prediction system designed for both research and forecasting applications. While I have yet to be versed in its capabilities, once I am, I will be able to use it to help solve the described problem in Section (1.2). Since a demonstration balloon flight is planned for the Mojave Desert in 2009, ideally we would be able to use wind fields from that location and time period. If this is not possible, we will use the wind fields in the same location, but from another time period (e.g. same time of year, but 2005).

3.2. **Discrete Mechanics and Optimal Control.** It is essential that I further my understanding of the DMOC method of calculating optimal trajectories. This will become necessary when analyzing results and dealing with issues that might arise. Without understanding the tools being used, I will not be able to interpret the results they produce.

DMOC was demonstrated for computing an optimal trajectory in a two-dimensional, time-independent flow. However, since the balloon is equipped with height control, it will be advantageous to determine these optimal trajectories in a three-dimensional wind field. Furthermore, this wind field will likely be variable over time and thus the optimal trajectory is unlikely to stay the same. Dealing with this issue will likely require discretizing the trajectory and then utilizing DMOC to compute the optimal trajectory between each discrete time step. Doing so will allow the balloon to adjust its path to the variable winds.

Most likely this will be first accomplished for a synthesized wind field. If this can be solved in this situation, then the problem can be extended for actual wind fields produced by the Weather Research and Forecasting model.

This method will likely perform best under the assumption that the wind field, though variable, will not make drastic, unexpected changes during the travel of the balloon. However, it will be important to investigate the robustness of the method if something unexpected takes place. It will be worthwhile to investigate what will happen in unexpected situations and speculate ways to overcome such problems.

Ultimately, however, it is unlikely that the balloon will have perfect information about the wind fields on Titan. As of right now, it is unclear exactly what information the balloon will be able to gather though we could hypothesize. This may recast the problem and make it much more difficult to solve. If time permits, I could investigate this issue as well.

3.3. **Functions of Many Variables.** I am currently investigating if an efficient tabulation of a function of many variables can be useful for this problem. There is a good chance that DMOC will be used to determine optimal trajectories for the balloon on Titan. An alternative would be to precompute all the different possible trajectories for a balloon based on a feasible set of maneuvers and then use a look-up table to determine the optimal trajectory in a rapid manner. It is my understanding that this type of approach was demonstrated for a helicopter flying in a canyon.

4. Appendix

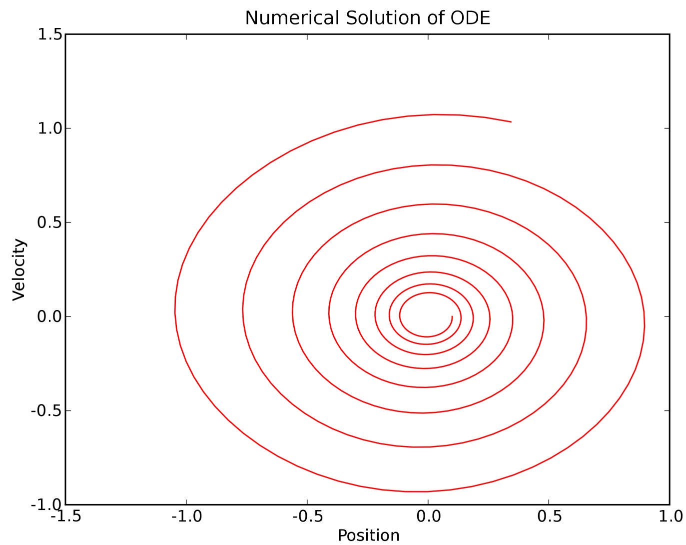

Figure 4.1. Explicit Euler Method for Pendulum

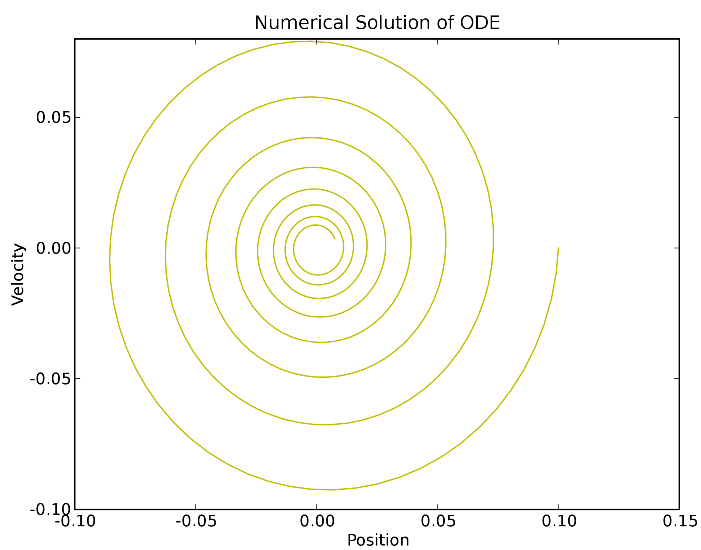

Figure 4.2. Implicit Euler Method for Pendulum

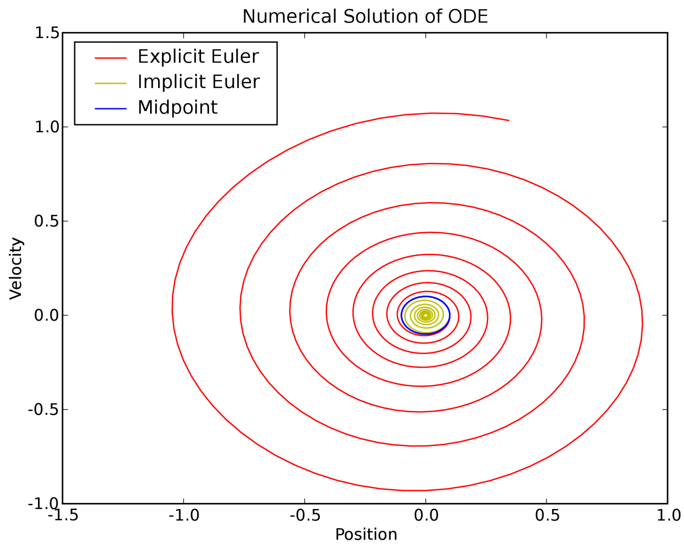

FIGURE 4.3. Comparison of Explicit and Implicit Euler and Implicit Midpoint Rule for Pendulum

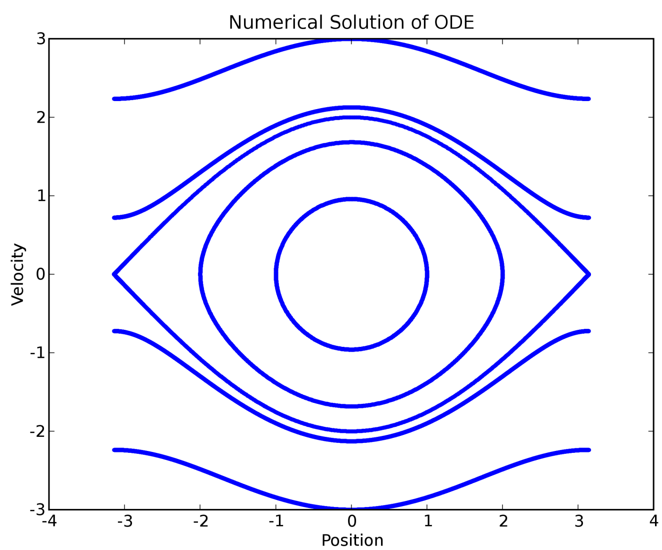

FIGURE 4.4. Phase Diagram for Pendulum Generated with Implicit Midpoint Rule

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