CALTECH
Control & Dynamical Systems

Geometry of the Full 2-Body Problem

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Full Body Workshop

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Context

- Example of a FBP: asteroid pairs.

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- Others will speak about dynamical systems aspects of these problems, including transport rates, etc.

Important Tools

- For mechanical systems with symmetry, some of the tools are:
  - ***Momentum maps***, **ie conserved quantities**
  - ***Reduction***, **shape space**
  - ***Stability and the energy-momentum method***
  - ***Geometric phases***

Restricted Problems

- ***Restricted*** means that one part of the system evolves in the field of another part of the system; Examples are
  - **Spherical pendulum on a Merry-Go-Round (the pendulum dynamics does not affect the rotation of the Merry-Go-Round)**
  - **Fluid flow on a rotating earth (the fluid does not affect the Earth's rotation)**
  - **Restricted 3-body problem**

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  - **Fluid flow on a rotating earth (the fluid does not affect the Earth's rotation)**
  - **Restricted 3-body problem**
- We typically handle restricted problems by the theory of ***moving systems***.

Restricted 3-body Problem

- consider the ***planar case***—the ***spatial case*** is similar
- ***Kinetic energy*** (wrt inertial frame) in rotating coordinates:

$$K(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right]$$

- ***Lagrangian*** is K.E. − P.E., given by

$$L(x, y, \dot{x}, \dot{y}) = K(x, y, \dot{x}, \dot{y}) - V(x, y); \quad V(x, y) = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

- ***Euler-Lagrange equations***:

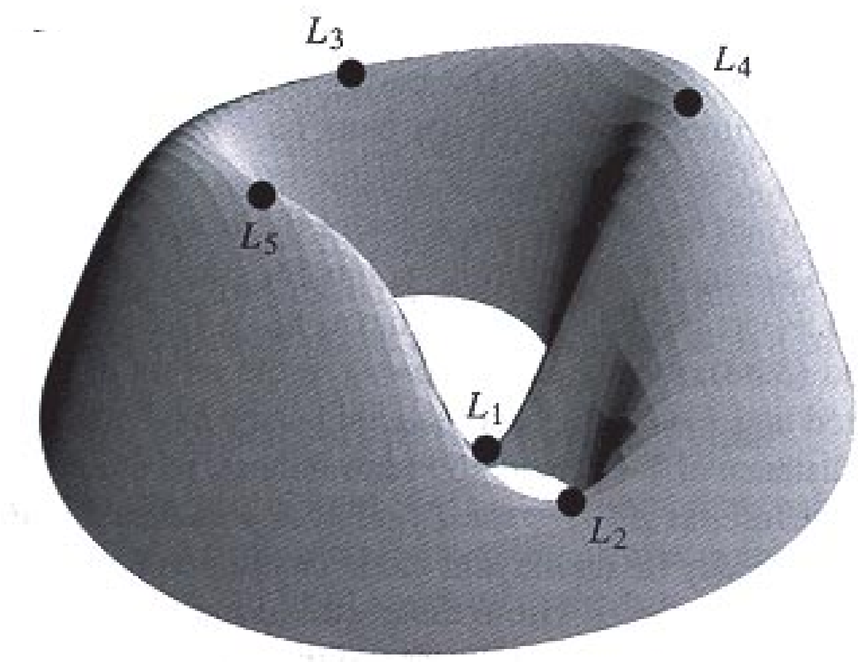
$$\boxed{\ddot{x} - 2\omega\dot{y} = -\frac{\partial V_\omega}{\partial x}, \qquad \ddot{y} + 2\omega\dot{x} = -\frac{\partial V_\omega}{\partial y}}$$

where the ***effective potential*** is

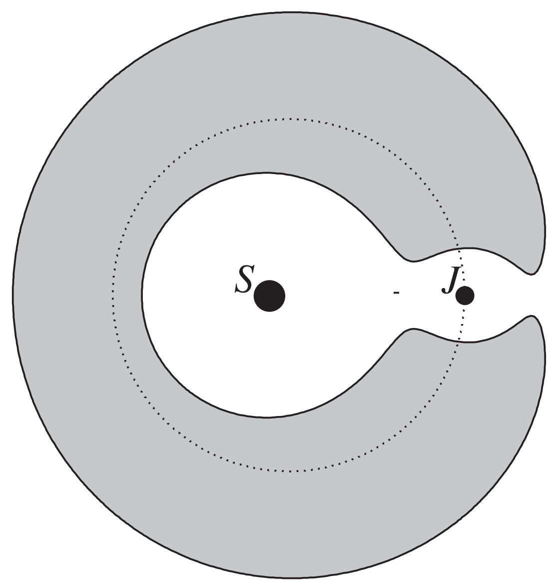
$$V_\omega = V - \frac{\omega^2(x^2 + y^2)}{2}$$

Effective potential

- Equations for the third body are those of a *particle moving in an effective potential plus a magnetic field* (Jacobi, Hill, etc)



Effective Potential



Level set shows the Hill region

More Tools of the Trade

- *Geometric mechanics* provides a general theory for (usually) nonrestricted problems: mechanical systems with symmetry. Eg, notions of amended potiential, relative equilibria, stability by the energy-momentum method, variational integration algorithms (symplectic integrators), etc.

F2BP

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- ***Configuration Manifold***: $Q = \mathrm{SE}(3) \times \mathrm{SE}(3)$
- The ***shape space*** Q/G gives the ***system shape*** and plays an important role in reduction theory.
- Lots of work by many people, as in Dan Scheeres talk

Reduction for the FBP

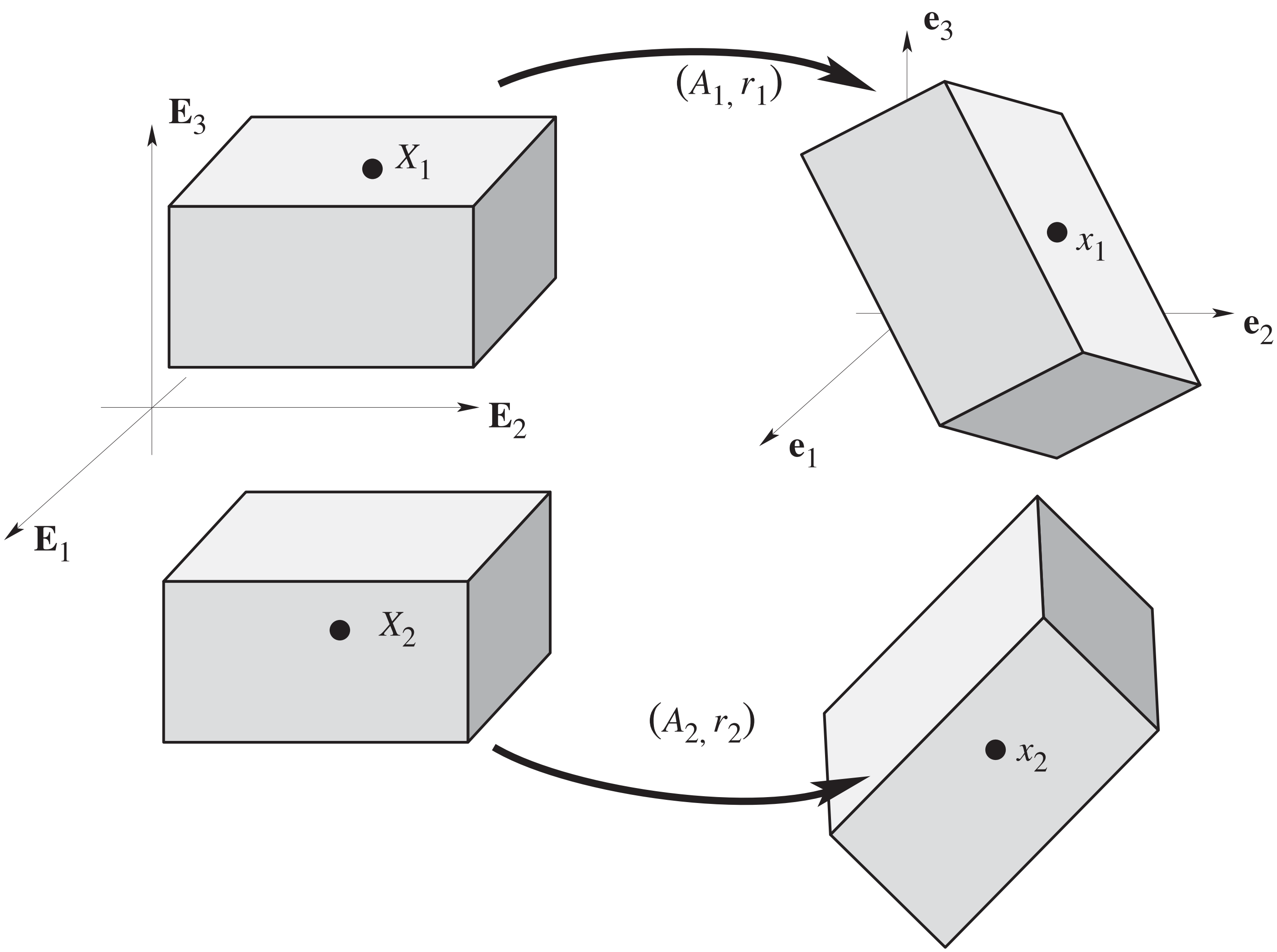
- ***Material points*** in a reference configuration X_i; $i = 1$ for body 1 and $i = 2$ for body 2
- Points in the current configuration x_i.
- Given a configuration

$$((A_1, r_1), (A_2, r_2)) \in \mathrm{SE}(3) \times \mathrm{SE}(3),$$

the ***material*** and ***spatial*** points are related by

$$x_1 = A_1 X_1 + r_1 \quad \text{and} \quad x_2 = A_2 X_2 + r_2$$

Reduction for the FBP



Reduction for the FBP

- *Lagrangian* equals kinetic minus potential energy:

$$
\begin{aligned}
&L(A_1, r_1, A_2, r_2, \dot{A}_1, \dot{r}_1, \dot{A}_2, \dot{r}_2) \\
&= \frac{1}{2}\int_{\mathcal{B}_1} \|\dot{x}_1\|^2 d\mu_1(X_1) + \frac{1}{2}\int_{\mathcal{B}_2} \|\dot{x}_2\|^2 d\mu_2(X_2) + \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{G d\mu_1(X_1) d\mu_2(X_2)}{\|x_1 - x_2\|} \\
&= \frac{m_1}{2}\|\dot{r}_1\|^2 + \frac{1}{2}\langle \Omega_1, I_1\Omega_1\rangle + \frac{m_2}{2}\|\dot{r}_2\|^2 + \frac{1}{2}\langle \Omega_2, I_2\Omega_2\rangle \\
&\qquad + \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{G d\mu_1(X_1) d\mu_2(X_2)}{\|A_1X_1 - A_2X_2 + r_1 - r_2\|}.
\end{aligned}
$$

- Here, for instance, $\Omega_1 = A_1^{-1}\dot{A}_1$ is the body angular velocity of the first body (with the usual identification of 3×3 skew matrices with vectors.

Reduction for the FBP

- **Goal:** Reduce by overall translations and rotations and bring the machinery of geometric mechanics to bear.
- SE(3) acts by the diagonal left action on Q:
  $(A, r) \cdot (A_1, r_1, A_2, r_2) = (AA_1, Ar_1 + r, AA_2, Ar_2 + r)$.
- ***Momentum map***
  $$\mathbf{J} : TQ \to \mathfrak{se}(3)^*$$
  is the ***total linear and angular momentum***.
- ***Shape space*** Q/G: one copy of SE(3); coordinatized by the ***relative attitude*** $A = A_1^{-1} A_2$ and ***relative position*** $R = A_2^T (r_1 - r_2)$.

Some Reduction Theory

- General reduction theory says that the variational principle and the equations of motion drop (in the appropriate sense) to the quotient space $(TQ)/G$. (Similarly to $(T^*Q)/G$ for Hamiltonian mechanics.

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- To get a nice realization of $(TQ)/G$, one chooses a ***connection*** $A : TQ \to \mathfrak{g}$ on the bundle $Q \to Q/G$ (assume that this is a principle bundle–i.e., there are no singularities.
- In this case, one gets a natural identification

$$\alpha_A : (TQ)/G \to T(Q/G) \times \tilde{\mathfrak{g}}$$

where $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ is the ***associated bundle***. Similarly for the Hamiltonian side of the story.

The FBP Case

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- First of all, ***shape space*** is given by

$$X = (G \times G)/G \cong G,$$

where the map $\pi : Q = G \times G \to G$ is given by

$$x = \pi(g_1, g_2) = g_1^{-1} g_2.$$

- A natural connection on the bundle $Q \to Q/G$ is given by

$$A(g_1\xi_1, g_2\xi_2) = \mathrm{Ad}_{g_2} \xi_2$$

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This connection has zero curvature.
- This has picked out one of the bodies as special.

The FBP Case

- This gives rise to the identification

$$\left(T(G \times G)\right)/G \cong G \times \mathfrak{g} \times \mathfrak{g}$$

where we map the class of $(g_1, \dot{g}_1, g_2, \dot{g}_2)$ to (x, w, ξ_2), where $x = g_1^{-1} g_2$ and $\xi_2 = g_2^{-1} \dot{g}_2$ as above and where $w = \dot{x} x^{-1}$.

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- Thus, by general theory, the equations of motion will reduce to equations for the variables (x, w, ξ_2).

More General Theory

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- The equations correspond to breaking up the variational principle into two parts: one for horizontal variations (***Lagrange*** part of the equations) and one for vertical variations (***Poincaré*** part of the equations).

Back to FBP

- One can work this all out quite explicitly for the general case of $Q = G \times G$, where, say, $G = \mathrm{SE}(3)$ and for Lagrangians of the form

$$L(g_1\xi_1, g_2\xi_2) = \frac{1}{2}\left[\mathrm{Tr}(K_1\xi_1^T\xi_1) + \mathrm{Tr}(K_2\xi_2^T\xi_2)\right] - V(g_1^{-1}g_2).$$

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- The reduced Lagrangian is given by $\ell(x, w, \xi_2)$ which equals the above expression, where $\xi_1 = -w + x\xi_2 x^{-1}$.

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- Similarly, one gets the Hamiltonian version of the equations, the reduced Poission structure, etc.

Systematic Structures

- For numerics as well as analysis of stability of relative equilibria (analog of the libration points), the variational and Hamiltonian structures are useful.
- Previous works ***guessed*** these structures and missed the variational structure altogether. Using reduction, one ***derives*** them in a simple and natural way, one gets the Jacobi integrals naturally, etc.
- Extra symmetries give extra conserved quantities and further reductions.

Restricted Simpler Case

- Restricted (as in restricted 3-body problem) simple case already exhibits the basic ejection and collision dynamics
- Point mass moving in the xy-plane under the gravitational field of a uniformly rotating elliptical body, without affecting its uniform rotation.

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- Restricted (as in restricted 3-body problem) simple case already exhibits the basic ejection and collision dynamics
- Point mass moving in the xy-plane under the gravitational field of a uniformly rotating elliptical body, without affecting its uniform rotation.
- ***Equations of motion*** relative to a rotating Cartesian coordinate frame and appropriately normalized:

$$\ddot{x} - 2\dot{y} = \frac{\partial V}{\partial x} \quad \text{and} \quad \ddot{y} + 2\dot{x} = \frac{\partial V}{\partial y},$$

Restricted Simpler Case

where

$$V(x,y) = \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{2}(x^2+y^2) + U_{22};$$

and where

$$U_{22} = \frac{3C_{22}\left(x^2-y^2\right)}{\left(x^2+y^2\right)^{5/2}}$$

- The coefficient C_{22} is the ***ellipticity***.
- ***Jacobi integral:*** $J = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - V$.

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- ***Jacobi integral:*** $J = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - V$.
- ***Moving systems approach*** gives, as in the RCTBP, the Lagrangian and Hamiltonian structure and Jacobi integral.

Restricted Simpler Case

- Lagrangian (kinetic minus potential energy) written in the rotating system and with angular velocity normalized to unity, is

$$L = \frac{1}{2}[(\dot{x} - y)^2 + (x + \dot{y})^2 + \dot{z}^2] - U(x, y, z).$$

where

$$U(x, y, z) = -\frac{1}{r} - U_{22}.$$

- Euler–Lagrange equations produce the previous equations and the Legendre transformation gives the Hamiltonian structure, the Jacobi integral, etc.

F2BP: Phase Space Structure

- The Jacobi integral (energy) is an indicator of the type of global dynamics possible.
- For energies above a threshold, $E > E_S$, corresponding to symmetric saddle points, movement between the ***realm*** near the asteroid (***interior realm***) and away from the asteroid (***exterior realm***) is possible. For energies $E \leq E_S$, no such movement is possible.

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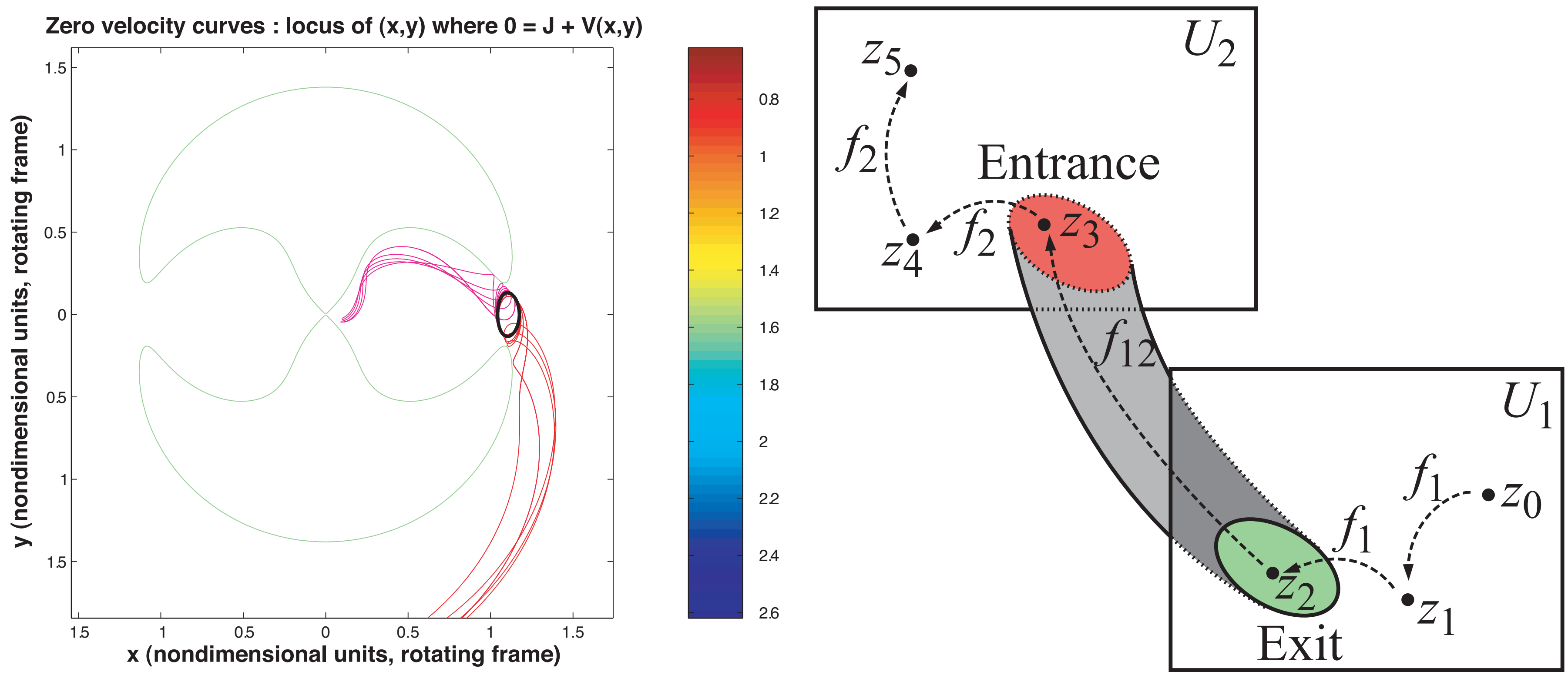
- As in the CRTBP, motion between realms is mediated by phase space ***tubes***.
- General theory allows us to transition what we learned in the CRTBP to this case.

F2BP: Phase Space Structure

- Phase space in each realm organized further into different ***resonance regions***, connected via ***lobes***.



(a) (b)

F2BP: Phase Space Structure

- Poincaré sections in the different realms, U_1 and U_2 are linked by tubes in the phase space. Under the Poincaré map f_1 on U_1, a trajectory reaches an ***exit***, the last Poincaré cut of a tube before it enters another realm. The map f_{12} takes points in the exit of U_1 to the ***entrance*** of U_2. The trajectory then evolves under the action of the Poincaré map f_2 on U_2.
- See Shane's talk and the material on the FBP website for further information.

Selected References

- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2003] ***Design of a Multiple Moon Orbiter***, *AAS/AIAA Space Flight Mechanics Meeting, Puerto Rico, Feb 9-12, 2003.*
- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2002] ***Constructing a low energy transfer between Jovian moons.*** *Contemporary Mathematics* **292, 129–145.**
- Gómez, G., W.S. Koon, M.W. Lo, J.E. Marsden, J. Masdemont and S.D. Ross [2001] ***Invariant manifolds, the spatial three-body problem and space mission design.*** *AAS/AIAA Astrodynamics Specialist Conference*.
- Koon, W.S., M.W. Lo, J.E. Marsden & S.D. Ross [2001] ***Resonance and capture of Jupiter comets.*** *Celestial Mechanics & Dynamical Astronomy* **81(1-2), 27–38.**
- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] ***Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics.*** *Chaos* **10(2), 427–469.**

For papers, movies, etc., visit **http://www.cds.caltech.edu/∼marsden** and **http://www.cds.caltech.edu/∼shane**

The End

TYPESETTING SOFTWARE: TEX, *Textures*, LATEX, `hyperref`, `texpower`, Adobe Acrobat 4.05
GRAPHICS SOFTWARE: Adobe Illustrator 10.0.1
LATEX SLIDE MACRO PACKAGES: Wendy McKay, Ross Moore