CDS 270-2: Lecture 6-3
Optimum Receiver Design for Estimation over Wireless Links
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Goals:

- To understand impact of wireless communication impairments on estimation over wireless
- To learn non-traditional designs for estimation over wireless applications
  - Optimization of packet drop
  - Use of cross-layer information paths
System Model (review)

Dynamical System $x(k)$

Observer $y(k)$

Transmitter

Wireless Channel

Receiver

Estimator $\hat{x}(k)$

node#1

node#2
To focus on communication noise, assume scalar quantities

Linear dynamical system:  \[ x(k + 1) = Ax(k) + w(k) \]

Observation:  \[ y(k) = Cx(k) + v(k) \]

\( w(k) \): Zero mean noise with variance of \( Q \)
\( v(k) \): Zero mean noise with variance of \( R \)
\( \hat{x}(k) \): Kalman filter estimate of \( x(k) \)
Wireless Transmission (review)

\[ \hat{y}(k) = y(k) + n(k) \]

- \( n(k) \) is communication noise with variance of \( \sigma_n^2(k) \)
Past Lectures

- Last week’s lectures: Only allow noise-free samples
- Last lecture:
  - We looked at a receiver that keeps all the packets and uses a cross-layer information path
  - We derived analytical expression to evaluate performance
  - We showed that the design is always stable
Today: Optimum Design

• What is the optimum packet drop for estimation and control over wireless links?
• What are the benefits of using channel knowledge in the estimator?
  – Stability & performance
• Consider general cases: general $\psi$ and $\sigma_n^2$ and system parameters
• Ideal noise profile: keeping only noise-free samples
  – Suitable for non delay-sensitive applications
Abstraction in the Higher Layer

\( \sigma_n^2(k) \) & \( p_{drop}(k) \):
- Function of TX/RX technologies like quantization, noise figure, modulation, channel coding, ..

Distribution of \( \psi_k \):
- Function of environment

\( \sigma_n^2(k) \) & \( p_{drop}(k) \):
Abstraction in the Higher Layer

\[ \sigma_n^2(k) \]

\[ p_{\text{drop}}(k) \]

\[ \psi_k \]

\[ \text{quantization noise} \]

\[ \text{Approximate} \]

\[ \psi_{\text{Thresh}} \]

\[ \sigma_n^2(k) \& p_{\text{drop}}(k) : \]

- Function of TX/RX technologies like quantization, noise figure, modulation, channel coding, ..

Distribution of \( \psi_k : \)

- Function of environment
## New Design Paradigms

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<th>Non-ideal noise profile</th>
<th>Ideal noise-profile</th>
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<td>Sinopoli et al. (TAC 04), to maintain stability:</td>
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<tr>
<td>drop</td>
<td>?</td>
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<tr>
<td>keep all</td>
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**Scenario#2:**

- Take impact of noisy samples into account
- Keep all noise-free samples

**Scenario#3:**

- Only keep noise-free samples

### Scenario#1: Sinopoli et al. (TAC 04), to maintain stability:

For non-mobile nodes: $p_{drop,scenario#1} < A^2$
Scenario #2: No Channel Info Available

- What is the optimum packet drop design?
- If $\psi_{Thresh}$ is too high, many packets are dropped => information loss
- If $\psi_{Thresh}$ is too low, estimation will be too noisy
- Intuitively, there should be an optimum $\psi_{Thresh}$
Scenario#2

- In Scenario#2, KF does not know anything about the quality of the communication link.
- From the point of view of the Kalman filter, the link was perfect.
- To focus on communication noise, we assume that observation noise is negligible in the derivations of scenario#2.
Scenario#2

- Then $\hat{x}(k + 1) = \begin{cases} A\hat{x}(k) & \text{if } k^{th} \text{ sample is dropped} \\ AC^{-1}\hat{y}(k) & \text{if } k^{th} \text{ sample is kept} \end{cases}$

- We will have the following recursion for estimation error variance:

$$P(k + 1) = A^2 P(k) + Q - \frac{A^2 P(k) - A^2 C^{-2}\sigma_n^2(\psi(k))}{S(k)}$$

where $S(k) = \begin{cases} 1 & \psi(k) \geq \psi_{\text{Thresh}} \\ \infty & \text{else} \end{cases}$
Scenario#2

• Averaging over Signal to Noise Ratio distribution, \( f(\psi) \), will result in the following:

\[
\overline{P}(k+1) = A^2 p_L(\psi_{Thresh}) \overline{P}(k) + A^2 C^{-2} p_N(\psi_{Thresh}) + Q
\]

where

\[
p_L(\psi_{Thresh}) = p_{drop} = \int_{0}^{\psi_{Thresh}} f(\psi) d\psi
\]

\[
p_N(\psi_{Thresh}) = \int_{\psi_{Thresh}}^{\infty} \sigma_n^2(\psi) f(\psi) d\psi
\]
Stability Condition for Scenario#2

To keep average estimation error variance bounded:

\[ P_{\text{drop}} = \int_{0}^{\psi_{\text{Thresh}}} f(\psi) d\psi < A^{-2} \]

Remark: Define **stability range** as the range of average Signal to Noise Ratios or \( A \) matrices for which estimation is stable. Having lower \( \psi_{\text{Thresh}} \) will increase the stability range.
Scenario#2: Optimum Performance

• **Theorem 1: Balance of Information Loss & Communication Noise**
  
  – Consider asymptotic average estimation error variance:
  
  \[ P(\infty) = \frac{A^2 C^{-2} p_N(\psi_{\text{Thresh}}) + Q}{1 - A^2 p_L(\psi_{\text{Thresh}})} \]
  
  \[ \text{for } p_L(\psi_{\text{Thresh}}) < A^{-2} \]
  
  – Optimum \( \psi_{\text{Thresh}} \) that minimizes asymptotic average estimation error variance will be as follows:

  \[ \psi_{\text{Thresh, opt}} = \begin{cases} \psi_{\text{Thresh}}^* & \psi_{\text{Thresh}}^* \geq 0 \\ 0 & \text{else} \end{cases} \]
Scenario#2: Optimum Performance

• Where $\psi^*_{\text{Thresh}}$ balances information loss and communication noise as follows:

$$p_L(\psi^*_{\text{Thresh}}) + p_{N,\text{normalized}}(\psi^*_{\text{Thresh}}) + \frac{C^2 Q}{A^2 \sigma_n^2 (\psi = \psi^*_{\text{Thresh}})} = A^{-2}$$

where

$$p_{N,\text{normalized}}(\psi^*_{\text{Thresh}}) = \frac{p_N(\psi^*_{\text{Thresh}})}{\sigma_n^2 (\psi = \psi^*_{\text{Thresh}})}$$

Eq. #1
Proof of Theorem 1

• Let $\psi_{\text{Thresh}}^*$ represent any solution to Eq#1. Let $\psi_{\text{Thresh}}^c$ represent the critical stability Threshold: $1 - A^2 p_L(\psi_{\text{Thresh}}^c) = 0$

• We have $\psi_{\text{Thresh}}^* < \psi_{\text{Thresh}}^c$

• It is easy to verify that
$$\frac{\partial P(\infty)}{\partial \psi_{\text{Thresh}}} = 0 \quad \text{at} \quad \psi_{\text{Thresh}} = \psi_{\text{Thresh}}^*$$

• We have to prove that Eq#1 has a unique solution
Proof of Theorem 1 (cont.)

Assume that Eq#1 has two solutions: \( \psi_{Thresh,1}^* \) and \( \psi_{Thresh,2}^* > \psi_{Thresh,1}^* \).

Since \( \sigma_n^2 \) is a non-increasing function of \( \psi \), we will have

\[
\begin{align*}
&\left[ p_L(\psi_{Thresh,1}^*) + p_{N, normalized}(\psi_{Thresh,1}^*) + \frac{C^2 Q}{A^2 \sigma_n^2(\psi = \psi_{Thresh,1}^*)} \right] - \\
&\left[ p_L(\psi_{Thresh,2}^*) + p_{N, normalized}(\psi_{Thresh,2}^*) + \frac{C^2 Q}{A^2 \sigma_n^2(\psi = \psi_{Thresh,2}^*)} \right] = \\
&\int_{\psi_{Thresh,1}^*}^{\psi_{Thresh,2}^*} f(\psi) d\psi + \int_{\psi_{Thresh,2}^*}^{\psi_{Thresh,1}^*} \frac{\sigma_n^2(\psi)f(\psi)}{\sigma_n^2(\psi = \psi_{Thresh,1}^*)} d\psi \\
&+ \left[ \frac{1}{\sigma_n^2(\psi = \psi_{Thresh,1}^*)} - \frac{1}{\sigma_n^2(\psi = \psi_{Thresh,2}^*)} \right] \int_{\psi_{Thresh,2}^*}^{\infty} \sigma_n^2(\psi)f(\psi) d\psi + \\
&\frac{C^2 Q}{A^2} \left[ \frac{1}{\sigma_n^2(\psi = \psi_{Thresh,1}^*)} - \frac{1}{\sigma_n^2(\psi = \psi_{Thresh,2}^*)} \right] < 0 \\
&\text{Therefore } \psi_{Thresh,1}^* = \psi_{Thresh,2}^*
\end{align*}
\]
Remarks on Optimum Performance

• Theorem 1 shows that as long as Eq#1 has a positive solution, then the optimum way of dropping packets is the one that balances loss of information and the amount of communication noise that enters the estimation process.

• If process noise is the dominant factor compared to the communication noise, Eq#1 may not have a positive solution. Then keeping all the packets is optimum.
Example: Optimum Packet Drop for Scenario#2

\[ A = 2 \]
\[ SNR_k : \ exp.\ dist. \]

Asymptotic average estimation error

\[ SNR = 5\, dB \]
\[ SNR = 10\, dB \]
\[ SNR = 15\, dB \]
\[ SNR = 30\, dB \]
\[ SNR = 20\, dB \]
\[ SNR = 25\, dB \]

Keep more

Drop more

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Scenario#3: Knowledge of channel available for KF

\[ P(k+1) = A^2 P(k) + Q - \frac{A^2 C^2 P^2(k)}{C^2 P(k) + \sigma^2_z(\psi(k))} \]

where

\[ \sigma^2_z(\psi(k)) = \begin{cases} \sigma^2_n(\psi(k)) + R & \psi(k) \geq \psi_{Thresh} \\ \infty & \text{else} \end{cases} \]
Scenario#3: Stability Condition

Lemma 1: Stability region of scenario#1 includes that of scenario#3

Proof: consider a special case of scenario#1 with $R=0$. Let $g(k)$ and $P(k)$ represent estimation error variances of scenario#1 with $R=0$ and scenario#3 respectively. We will have,

$$\quad g(k+1) = A^2 p_L \bar{g}(k) + Q$$
Scenario #3: Stability Condition

\[ P(k + 1) \geq A^2 P(k) + Q - \frac{A^2 C^2 P^2(k)}{C^2 P(k) + S(k)} \]

where \( S(k) = \begin{cases} 0 & \psi(k) \geq \psi_{Thresh} \\ \infty & \text{else} \end{cases} \)

We will have \( \overline{P}(k + 1) \geq A^2 P_L \overline{P}(k) + Q \).

Then, if \( \overline{P}(k) \geq \underline{g}(k) \Rightarrow \overline{P}(k + 1) \geq \underline{g}(k + 1) \)
Scenario#3: Stability Condition

Lemma 2: Stability region of scenario#3 includes that of scenario#2

Proof: Let \( q(k) \) represent estimation error variance of scenario#2 with \( R \neq 0 \), where no knowledge of \( R \) is available at the KF. We will have,

\[
q(k + 1) = A^2 p_L q(k) + Q + A^2 C^{-2} p_{N,R}
\]

where \( p_{N,R} = p_N + (1 - p_L) R \)
Scenario#3: Stability Condition

\[ E(P(k+1) \mid P(k)) = \]
\[ (1 - p_L) E(P(k+1) \mid P(k), \psi(k) > \psi_{Thresh}) + \]
\[ p_L E(P(k+1) \mid P(k), \psi(k) < \psi_{Thresh}) \]

\( P(k+1) \) is a concave function of \( \sigma^2_z \).
Then using conditional Jensen's Inequality,

\[ E(P(k+1) \mid P(k), \psi(k) > \psi_{Thresh}) \leq \]
\[ A^2 P(k) + Q - \frac{A^2 C^2 P^2(k)}{C^2 P(k) + E(\sigma^2_z(\psi(k)) \mid \psi(k) > \psi_{Thresh})} \]
Scenario#3: Stability Condition

Then,

\[
E(P(k+1) \mid P(k)) \leq A^2 P(k) + Q + \frac{(p_L - 1)A^2 C^2 P^2(k)}{C^2 P(k) + E(\sigma^2_z(\psi(k)) \mid \psi(k) > \psi_{\text{Thresh}})}
\]

The third term on the right hand side is a concave function of \( P(k) \).

Applying Jensen's Inequality,

\[
E(P(k+1)) \leq A^2 E(P(k)) + Q + \frac{(p_L - 1)A^2 C^2 E^2(P(k))}{C^2 E(P(k)) + E(\sigma^2_z(\psi(k)) \mid \psi(k) > \psi_{\text{Thresh}})}
\]

Noting that \( E(\sigma^2_z(\psi(k)) \mid \psi(k) > \psi_{\text{Thresh}}) = \frac{p_{N,R}}{1 - p_L} \Rightarrow \)

if \( E(P(k)) \leq E(q(k)) \Rightarrow E(P(k+1)) \leq E(q(k+1)) \)

\[\blacksquare\]
Scenario#3: Cross-layer Design

• **Theorem 2**: Cross-layer path on channel quality does **NOT** impact stability region
  – Proof: Lemma 1 and Lemma 2 proved that stability region of scenario#1 includes that of scenario#3 and stability region of scenario#3 includes that of scenario#2. We proved that stability region of scenario#2 is the same as scenario#1. Therefore, scenario#3 will have the same stability condition.

• Keeping all packets minimizes average estimation error variance
  – Easy to prove: see CDC05 in the reference list

• Cross-layer available => keep all packets for stability & performance
Effect of Cross-Layer Design

Solid: no cross-layer
Dashed: cross-layer

Average estimation error at $k=300$

$\text{SNR}_{\text{ave}} = 5\,\text{dB}$
$\text{SNR}_{\text{ave}} = 10\,\text{dB}$
$\text{SNR}_{\text{ave}} = 15\,\text{dB}$
$\text{SNR}_{\text{ave}} = 20\,\text{dB}$
$\text{SNR}_{\text{ave}} = 25\,\text{dB}$
$\text{SNR}_{\text{ave}} = 30\,\text{dB}$

$\text{SNR}_{\text{Thresh}}$
Estimation Over Wireless: Summary

• We studied optimum packet drop mechanism
• We proved that stability condition is the SAME independent of cross-layer or shape of communication noise variance

• Cross-layer on channel knowledge available:
  – keep packets for both stability & performance

• Cross-layer on channel knowledge not available:
  – Stability range main factor: keep packets
  – Estimation error main factor: packet drop to balance information loss and comm. noise
Possible Projects: Study Impact of Communication Impairments on Estimation and Control over Wireless

- Have a node estimate/control a dynamical system over a wireless link
- study the impact of channel variations, channel correlation
- explore redesigning the communication side: keeping packets, using cross-layer design, control packet drop, use of SNR info in estimation/control
- Try other communication protocols like analog communication, sensor network protocols like 802.15.4 (ZigBee),
- Compare different protocols (ZigBee, 802.11b, Bluetooth, Analog, …)
- Survey of suitable networking protocols for these applications