\[ \dot{x} = f(x), \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Consider an equilibrium point \( x = x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \)

Let \( z = f(z_1, z_2) \)

From equilibrium, use Taylor series about \( z = 0 \), focus on each \( z \); individually:

\[
\Delta z_1 = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} + o(z_1^2, z_2^2)
\]

Similarly,

\[
\Delta z_2 = \begin{bmatrix} \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_1} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} + o(z_1^2, z_2^2)
\]

Stack them:

\[
\begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} + \text{HOT}
\]

How disturbance propagates Jacobian \( (A) \) about \( 0 \) small disturbance \( \Delta x \) \( \Rightarrow z \)

Linearization about equilibrium given by

\[
\dot{x} = Ax, \quad x = x^*
\]
Now, given a nonlinear system:

\[ \dot{x} = f(x, u) \]
\[ y = h(x, u) \]

Can define all the matrices necessary to turn this into a linear system:

\[ x = Ax + Bu \]
\[ y = Cx + Du \]

By defining them analogously to the Jacobian:

Take 2D, single input:

\[ A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix} \]

\[ C = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} \quad D = \begin{bmatrix} \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial u} \end{bmatrix} \]

For multiple controls, \( B \) \& \( D \) can have more columns (one column for each control).

Behavior of the linearization may tell you useful info about nonlinear system:

- \( A \) stable \( \Rightarrow \) full system is locally stable
- \( A \) unstable \( \Rightarrow \) full system unstable
- \( A \) has mixed stability \( \Rightarrow \) can't say (e.g. saddle point)
Discrete Time Example

Predator/Prey

\[ H_{k+1} = H_k + b H_k - a L_k H_k = f_1 \]
\[ L_{k+1} = L_k - d L_k + e L_k H_k = f_2 \]

Jacobian: \( \frac{\partial f_1}{\partial H} = 1 + b - a L_k \)
\[ \Rightarrow \frac{\partial f_1}{\partial L} = -a H_k \]
\[ \frac{\partial f_2}{\partial H} = e L_k \]
\[ \frac{\partial f_2}{\partial L} = 1 - d + e H_k \]

\[ A = \begin{bmatrix} 1+b & -a H_k \\ -a L_k & -d + e H_k \end{bmatrix} \]

Equilibrium Point:

\[ H_{k+1} = H_k \Rightarrow b H_k^* - a L_k^* H_k^* = 0 \]
\[ \Rightarrow L_k^* = \frac{b}{a} \]
\[ H_{k+1} = L_k \Rightarrow H^* = \frac{d}{e} \]
so the of point linearization is
\[
A|_{x_0} = \begin{bmatrix}
1 + \frac{b}{a} & -\frac{c}{a} \\
\frac{bc}{a} & 1 - \frac{d}{a}
\end{bmatrix}
\]

question: is this stable?

for a discrete-time system to be stable, we need
\[x_{k+1} < x_k \Rightarrow \text{eigenvalues of } A < 1\]
(Compare to continuous case, where we needed \[x < 0 \Rightarrow \text{eigenvalues of } A < 0\])

numerical values:
\[a = c = 0.014, \quad b = 0.6, \quad d = 0.7\]

\[
A|_{x_0} = \begin{bmatrix}
1.58 & -0.74 \\
0.6 & 0.3
\end{bmatrix}
\]

\[(1-x)^2 + 0.42\]
\[= 1 - 2x + x^2 + 0.42\]
\[= \lambda^2 - 2\lambda + 1.42 = 1 \pm \sqrt{-0.42}\]
unstable
two methods: - diagonalization
- express $x_0$ in the basis of eigenvectors

Harder example

HW #4 \((A, M, \Gamma, \delta_{eq}, 5:10)\)

Linear discrete time system:

\[ x_{k+1} = Ax_k + Bu_k \]
\[ y_k = Cx_k + Du_k \]

a) Find general form of output $y_k$ (show it's equal to...)

To find $y_k$, need $x_k$

Keep writing until you can see a recursion

\[ x_1 = Ax_0 + Bu_0 \]
\[ x_2 = Ax_1 + Bu_1 = A(Ax_0 + Bu_0) + Bu_1 \]
\[ x_3 = Ax_2 + Bu_2 = A(A(Ax_0 + Bu_0) + Bu_1) + Bu_2 \]
\[ x_4 = Ax_3 + Bu_3 = A(A(A(Ax_0 + Bu_0) + Bu_1) + Bu_2) + Bu_3 \]

Pop out the IC response: $A^k x_0$

Write out the control response to make it clearer

\[ x_{k+1} \xrightarrow{k-1} A^3 Bu_0 + A^2 Bu_1 + ABu_2 + Bu_3 \]

Highest power is $k-1$

So write $\sum_{i=0}^{k-1} A^{k-1-i} Bu_i$

Final form: $x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} Bu_i$
but we want output \( y_k = Cx_k + D u_k \)

\[
y_k = C \left[ A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i \right] + D u_k
\]

\[
= CA^k x_0 + \sum_{i=0}^{k-1} CA^{k-i-1} B u_i + D u_k
\]

b) show asymptotic stability for eigenvalues of \( A < 1 \)

**NOTE**: can unbound response WLOG (e.g. shift)

**Method 1**

**Diagonalization**

\[
A = T A^d T^{-1}, \quad A^d = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_n
\end{bmatrix}
\]

assumption of full basis \( T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \)

of eigenvectors allows this diagonalization

then \( x_{k+1} = \left( T A^d T^{-1} \right)^k x_0 \)

\[
= T (A^d)^k T^{-1} x_0
\]

\[
= T \begin{bmatrix}
\lambda_1^k & 0 \\
0 & \lambda_n^k
\end{bmatrix} T^{-1} x_0
\]

if \( \lambda_1 < 0 \), then \( \lim_{k \to \infty} x_{k+1} = 0 \)
method 2

express initial condition as a linear combination of eigenvectors.
(can do this because the basis of eigenvectors is full ⇒ span the state space)

\[ x_0 = \sum \alpha_i v_i \]

\[ \Rightarrow x_{k+1} = A^k x_0 \]

\[ = A^k \sum \alpha_i v_i \]

\[ = \sum \alpha_i A^k v_i \]

but recall the definition of eigenvector/eigenvalue pair

\[ A v_i = \lambda_i v_i \]

\[ x_{k+1} = \sum \alpha_i \lambda_i^k v_i \]

for \( \lambda_i < 0 \), \( \lim_{k \to \infty} x_{k+1} = 0 \)
c) $u_k = \sin(\omega k) \implies \text{WLOG } u_k = e^{j\omega k}$

Linear systems preserve frequency but may shift phase in steady state.

Let $x_k = Ne^{j\omega k}$

$y_k = Me^{j\phi}e^{j\omega k}$

Then from the state eqn:

$x_{k+1} = \frac{1}{N}Ne^{j\omega(k+1)} = ANe^{j\omega}e^{j\omega k} + Be^{j\omega k}$

$N e^{j\omega} e^{j\omega} = AN e^{j\omega} + B$

$(e^{j\omega I} - A)Ne^{j\omega} = B$

$Ne^{j\omega} = (e^{j\omega I} - A)^{-1}B$

$y_k = Me^{j\phi}e^{j\omega k} = CNe^{j\omega}e^{j\omega k} + De^{j\omega k}$

$= C(e^{j\omega I} - A)^{-1}B + D$