CDS 270-2: Lecture 6-2
Impact of Communication Noise on Estimation over Wireless Links

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Goals:
- To understand the impact of noisy wireless communication links on estimation over wireless
- To evaluate the performance of Kalman filtering over noisy mobile links
Reading for May 3\textsuperscript{th} and 5\textsuperscript{th}
Can also be found at
http://www.cds.caltech.edu/~yasi/

• Y. Mostofi and R. Murray, “Receiver Design Principles for Estimation over Fading Channels," Proceedings of Conference on Decision and Control (CDC), December 2005


Networked Sensing, Estimation & Control

Today: Modeling and impact of wireless links
System Model

Dynamical System \( x(k) \) → Observer \( y(k) \) → Transmitter → Wireless Channel → Receiver → Estimator \( \hat{x}(k) \)

node#1

node#2
System Model

To focus on communication noise, assume scalar quantities

Linear dynamical system:  \[ x(k + 1) = Ax(k) + w(k) \]

Observation:  \[ y(k) = Cx(k) + v(k) \]

\( w(k) \): Zero mean noise with variance of  \( Q \)
\( v(k) \): Zero mean noise with variance of  \( R \)
\( \hat{x}(k) \): Kalman filter estimate of  \( x(k) \)
Wireless Transmission

Past 4 classes: Studied cases where only noise-free samples came through
Error Detection & Packet Drop

• Drop criteria changes depending on the application

• Voice applications are delay-sensitive:
  – Calls are dropped only if crucial bits are corrupted
  – There is error detection for crucial bits
  – The rest of the error is either corrected or tolerated

• Data applications are not delay-sensitive:
  – Packets are only kept if no error is detected

• What is optimum for control over wireless? (we will get to this later)
Wireless Communication

• Impairments:
  – Signal attenuation
  – Multipath, fading & shadowing
  – Time-varying links
  – Limited bandwidth
  – Collision

• One measure of link quality:
  – **Received Signal to Noise Ratio** = Ratio of received signal power to receiver noise power
**Communication Noise**

- Impairments result in noisy reception:

\[ \hat{y}(k) = y(k) + n(k) = Cx(k) + v(k) + n(k) \]

- \( n(k) \) is communication noise with variance of \( \sigma_n^2(k) \)

- \( \sigma_n^2(k) \) is a function of received Signal to Noise Ratio
- Communication noise appears as additional observation noise
- This model provides the right abstraction for estimation and control
Communication Noise Variance

$\sigma_n^2(k)$ = communication noise variance

- Function of TX/RX technologies like quantization, noise figure, modulation, channel coding, ...

Distribution of $\psi_k$:
- Function of environment
- Common outdoor model: exponential distribution

$\psi_k$: Received Signal to Noise Ratio at $k^{th}$ transmission
Can the Estimator Know the Variance of Communication Noise?

- KF relies on using covariance of the observation noise
- In order for the estimator to know the quality of the communication link, a cross-layer information path is needed
- In general such paths can improve the performance considerably and have gotten considerable interest recently
- However, one has to be cautious since careless use of such paths can ruin the robustness of the system
Design Choices

- Estimation & control over wireless links are new applications
  - Need new design paradigms
- Possible design choices:
  - Receiver can keep all the samples
  - Receiver can optimize the packet drop
  - Cross-layer: estimator can use link quality information
- Consider *unstable processes*. We are interested in:
  - Analytical expressions to evaluate the performance of Kalman filtering over wireless noisy links
  - Optimizing packet drop (Friday lecture)
  - Stability condition (Friday lecture)
Performance Evaluation Example

• Consider the following example:
  – Receiver that keeps all the packets
  – KF that uses knowledge of channel quality (trust coefficient)
  – One class of channels:
    \[ \sigma_n^2(k) = \frac{\beta}{\psi_k} \text{ for } \beta > 0 \]
    – Exponential distributed \( \psi_k \)
    – Channel gets uncorrelated from one sample to next
  – In order to focus on communication impact, assume the following for this example: \( R = 0, Q = 0 & C = 1 \)
Performance Evaluation

\[ P_k = \left[ x(k) - \hat{x}(k) \right]^2 |_{\psi_{k-1}, \ldots, \psi_0} \]

We are interested in finding \( P_k \)

\[ P_{k+1} = \frac{A^2 \beta \times P_k}{\beta + \psi_k \times P_k} \]

\( P_{k+1,i} \) : average of \( P_{k+1} \) over \( \psi_k, \psi_{k-1}, \ldots, \psi_{k-i} \)

\[ P_{k+1,0} = E\left( \frac{A^2 \beta P_k}{\beta + \psi_k P_k} \mid P_k \right) = A^2 \Gamma \beta P_k \int_0^\infty \frac{e^{-\Gamma \psi_k}}{\beta + \psi_k P_k} d\psi_k = A^2 \Gamma \beta \times \Pi\left( \frac{\Gamma \beta}{P_k} \right) \]

with \( \Pi(z) = e^z \text{Expint}(z) \), where \( \text{Expint}(z) = \int_z^\infty \frac{e^{-t}}{t} dt \)

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Performance Evaluation (cont.)

\[
P_{k+1,0} = A^2 \Gamma \beta \times \Pi \left( \frac{\Gamma \beta}{P_k} \right) = A^2 \Gamma \beta \times \Pi \left( \frac{\Gamma (\beta + \psi_{k-1} P_{k-1})}{A^2 P_{k-1}} \right)
\]

Lemma 1: Consider an exponentially dist. \( \psi \) with

\[
\Gamma = \frac{1}{\psi}
\]

We will have the following for an arbitrary \( q > 0 \) and \( A > 1 \):

\[
\frac{\Pi \left( \frac{\Gamma \beta}{q \times A^{2i}} \right)}{\Pi \left( \frac{\Gamma (\beta + q \psi)}{q \times A^{2i}} \right)} = \frac{\Pi \left( \frac{\Gamma \beta}{q} \right)}{\Pi \left( \frac{\Gamma \beta}{1 - A^{-2i}} \right)} - \frac{\Pi \left( \frac{\Gamma \beta}{1 - A^{-2i}} \right)}{\Pi \left( \frac{\Gamma \beta}{q} \right)} \quad i \geq 1
\]
Performance Evaluation (cont.)

Using Lemma 1:

\[
P_{k+1,1} = A^2 \Gamma \beta \left[ \Pi \left( \frac{\Gamma \beta}{A^2 P_{k-1}} \right) \frac{1}{1 - A^{-2}} - \frac{\Pi \left( \frac{\Gamma \beta}{P_{k-1}} \right)}{1 - A^{-2}} \right]
\]

Similarly:

\[
P_{k+1,m} = \sum_{z=0}^{m} B_{z,m} \Pi \left( \frac{\Gamma \beta}{A^{2z} P_{k-m}} \right),
\]

where \( B_{0,0} = A^2 \Gamma \beta \). The goal is to find \( B_{z,m=k} \).

Let \( T_k(z,m) = \Pi \left( \frac{\Gamma \beta}{A^{2z} P_{k-m}} \right) \). Then

\[
P_{k+1,m} = \sum_{z=0}^{m} B_{z,m} T_k(z,m).
\]
Performance Evaluation (cont.)

\[ P_{k+1,m} = \sum_{z=0}^{m} B_{z,m} T_k (z, m) \]

Substituting \( P_{k-m} \) as a function of \( P_{k-m-1} \) and averaging over \( \psi_{k-m-1} \) will result in the following for \(-1 \leq m \leq k - 1\) (using Lemma 1):

\[ P_{k+1,m+1} = \sum_{z=0}^{m} \frac{B_{z,m}}{\xi_{z+1}} T_k (z + 1, m + 1) - \sum_{z=0}^{m} \frac{B_{z,m}}{\xi_{z+1}} T_k (0, m + 1) \]

\[ = \sum_{i=1}^{m+1} \frac{B_{i-1,m}}{\xi_{i}} T_k (i, m + 1) - \left[ \sum_{z=0}^{m} \frac{B_{z,m}}{\xi_{z+1}} \right] T_k (0, m + 1) \]

\[ = \sum_{i=0}^{m+1} B_{i,m+1} T_k (i, m + 1) \text{ where } \xi_{i} = 1 - \frac{1}{A^{2i}} \]
Performance Evaluation

• Finally

$$P_{k+1} = \sum_{i=0}^{k} B_{i,k} e^{A^{2i}P_0} \text{Expint}(\frac{\Gamma \beta}{A^{2i}P_0})$$

$$B_{i,k} = \begin{cases} 
- \sum_{z=0}^{k-1} \frac{B_{z,k-1}}{\xi_{z+1}} & i = 0 \\
\frac{B_{i-1,k-1}}{\xi_i} & i \neq 0 
\end{cases}
$$

$$\Gamma = \frac{1}{\psi}$$

$$B_{0,0} = A^{2\Gamma}$$

$$\xi_i = 1 - \frac{1}{A^{2i}}$$
Stability Condition

- $P_k$ will be bounded as long as $\psi \neq 0$

Proof: $P_{k+1} = \frac{A^2 \beta \times P_k}{\beta + \psi_k P_k}$, $P_{k+1}$ is a concave function of $P_k$.

Using Jensen's inequality:

$$P_{k+1} = E_{\psi_k} \left( E_{P_k} (P_{k+1}) \right) \leq E_{\psi_k} \left( \frac{A^2 \beta \times P_k}{\beta + \psi_k P_k} \right)$$

$$= A^2 \beta \Gamma \times e^{\Gamma \beta / P_k} \text{Expint} \left( \frac{\Gamma \beta}{P_k} \right)$$

If $\frac{P_k}{\mu_0} > \frac{\Gamma \beta}{\mu_0}$ $\Rightarrow$ $P_{k+1} < P_k$ where $A^2 \mu_0 e^{\mu_0} \text{Expint}(\mu_0) = 1$
Performance Evaluation

\[ \overline{P_k} = \beta A \]

\[ A = 4 \]
\[ \beta = 0.1 \]
Performance Evaluation (cont.)

Keeping All packets

\[
\begin{align*}
\text{dBSNRthresh} = 40 \text{dB} \\
\text{dBSNRthresh} = 30 \text{dB} \\
\text{dBSNRthresh} = 20 \text{dB} \\
\text{dBSNRthresh} = 10 \text{dB} \\
\text{dBSNRthresh} = 0 \text{dB} \\
\end{align*}
\]
Summary (so far)

• We looked at a receiver that keeps all the packets and uses a cross-layer information path
• We derived analytical expression to evaluate performance
• We showed that the design is always stable
Possible Extensions

• Derive average estimation error variance for:
  – Other communication noise variances
  – General noise variance
  – General Signal to Noise Ratio distribution
  – Vector case

• Derive expressions for other moments of estimation error variance
Wireless Transmission

\[ \hat{y}(k) = y(k) + n(k) \]

\( n(k) \) is communication noise with variance of \( \sigma_n^2(k) \)
Optimum Design

• What is the optimum packet drop for estimation and control over wireless links?

• What are the benefits of using channel knowledge in the estimator?
  – Stability & performance

• Consider general cases: general $\psi$ and $\sigma_n^2$ and system parameters

• Ideal noise profile: keeping only noise-free samples
  – Suitable for non delay-sensitive applications
Abstraction in the Higher Layer

\[ \sigma_n^2(k) \]

- Function of TX/RX technologies like quantization, noise figure, modulation, channel coding, ..

\[ p_{\text{drop}}(k) \]

- Distribution of \( \psi_k \):
  - Function of environment
Abstraction in the Higher Layer

Function of TX/RX technologies like quantization, noise figure, modulation, channel coding, ..

\[ \sigma_n^2(k) \] & \[ p_{\text{drop}}(k) \] :
- Function of TX/RX technologies like quantization, noise figure, modulation, channel coding, ..

Distribution of \( \psi_k \):
- Function of environment
## New Design Paradigms

### Non-ideal noise profile

<table>
<thead>
<tr>
<th></th>
<th>cross-layer</th>
<th>no cross-layer</th>
<th>Ideal noise-profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>drop</td>
<td>Scenario#3</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>keep all</td>
<td>?</td>
<td>?</td>
<td>Scenario#1 Sinopoli et al. &amp; Liu et al.</td>
</tr>
</tbody>
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### Scenario#1: Sinopoli et al. (TAC 04), to maintain stability:

for non-mobile nodes: \( p_{\text{drop,scenario#1}} < A^{-2} \)
Next Class

- We will complete the table next time for general communication noise variance and Signal to Noise Ratio distribution