IACATU

CD 110a Midterm Review

1. Overview

- Central control systems ideas:
  - Feedback, input-output behavior of systems and dynamics
    a) Dynamics: change over time
    b) Input-output: \( U \rightarrow \text{system} \rightarrow Y \) (Input) (Output)
      
      Example of a dynamic input-output system:
      A linear ODE \( y'' + 2y' + y = u + u' \)
      (can write output explicitly: \( y = -y'' - 2y' + u + u' \))

    c) Feedback
      
      Benefits of feedback:
      - Resilience to external disturbance
      - Resilience to internal variation
      * Baggage result: can get linear behavior from variable, nonlinear components!

    \( \begin{array}{cc}
    \text{sys1} \\
    \text{sys2}
    \end{array} \)

    \( \Rightarrow \) No cause & effect,
    \( \Rightarrow \) Must consider system as a whole

2. A Control System:
   
   Bases correcting actions on the difference between the actual and desired performance.

3. The course so far has mostly been concerned so far with dynamic behavior, with a little bit of feedback/Io
   - Equilibrium points
   - Stability of equilibrium points
   - A little bit of state feedback
II. Modeling and dynamics

1. Usually - No blocks have hidden state that is not directly measurable.

   Example: another ODE: \( x'' + x = u \)
   \[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u \rightarrow \text{sys} \rightarrow y \]

   (don't know \( x \) from output)

2. State space form
   - State is set of vars that characterize motion of system
   - State space is set of all possible states

   - Form: \( \dot{x} = f(x, u) \) \quad \begin{bmatrix} x \in \mathbb{R}^n \end{bmatrix} \) (state)
   \( y = h(x, u) \) \quad \begin{bmatrix} y \in \mathbb{R}^m \end{bmatrix} \) (output)
   \( u \in \mathbb{R}^p \) (input)

   (full, nonlinear case)

   Example: ODE \( \rightarrow \) S.S.

   \[ \ddot{z} = z - z^2 \]
   \[ \begin{array}{l}
   \text{set} \quad x_1 = z \\
   x_2 = \dot{z} \\
   \frac{dx_1}{dt} = x_2 \\
   \frac{dx_2}{dt} = x_2 \end{array} \]

   \[ \frac{dx}{dt} = \begin{bmatrix} x_2 \\ x_1-x_2^2 \end{bmatrix} \]
III. Extended example
- equilibrium points
- stability

Problem:
Consider the system: (similar to Exercise 3.6 in book)

\[ \begin{align*}
\dot{x}_1 &= a(x_1 - x_2) \\
\dot{x}_2 &= b(x_2 - x_1) - c \frac{\dot{x}_2}{\dot{x}_1 + x_2} + e + u^3 = f(x_1, u) \\
\end{align*} \]

where \( a, b, c, d, e > 0 \), \( c = b \left( \frac{d+1}{d} \right)^2 \)

Find an equilibrium point, analyze stability, and if unstable, use state feedback to stabilize.

Solution:

**equilibrium points at** \( \dot{x}_1 = \dot{x}_2 = 0 \) (assuming \( u = 0 \))

1. \( \dot{x}_1 = 0 = a(x_1 - x_2) \Rightarrow x_1 = x_2 \)
2. \( \dot{x}_2 = 0 = -c \frac{\dot{x}_2}{\dot{x}_1 + x_2} + e \Rightarrow x_2 \left( \frac{e}{c} \right) + \frac{e d}{c} = 1 \Rightarrow x_2 = \frac{ed}{c} \left( 1 - \frac{e}{c} \right) \Rightarrow x_1 = 1 \)

only eq. point \( x_e = (1, 1) \)

**analysis of stability at** \( (1,1) = x_e \)
- linearize to find behavior of linear system near \( x_e \).
Linearization

Taylor series: \( f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \ldots \)

n-d \( x \) vector: \( f(\hat{x}) \approx f(a) + \frac{\partial F(\hat{a})}{\partial x}(\hat{x}-a) + \ldots \)

where \( \frac{\partial F}{\partial x} = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \\ \frac{\partial F}{\partial x_n} & \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_{n-1}} \end{bmatrix} \)

the two terms \( f(x) \approx f(a) + \frac{\partial F}{\partial x}(a)(x-a) \)
are the linearization of the dynamics.

In controls, we often want a system to get somewhere and stop.
This only happens at equilibrium points

\[ \dot{x} = f(x) \quad \text{where} \quad \dot{x} = 0 \Rightarrow \text{where} \quad f(x_e) = 0, \]

You can push a system to a desired equilibrium point with proper baseline input. But for now, we're going to assume we're already at the desired equilib. point, we just want to make sure it's stable.

The linearization becomes:

\[ f(x) \approx f(x_e) + \frac{\partial F}{\partial x}(x-e) (x-x_e) \]

or \( \dot{z} = A(z-x_e) \quad \text{where} \quad A = \frac{\partial F}{\partial x}\big|_{x=x_e} \)

Selecting \( z = x-x_e \), \( \dot{z} = Az \)

Differentiating, \( \dot{z} = Ax \)

Similarly, \( B = \frac{\partial h}{\partial u}\big|_{x=x_e,u=u_e} \), \( C = \frac{\partial h}{\partial x}\big|_{x=x_e,u=u_e} \), \( D = \frac{\partial h}{\partial u}\big|_{x=x_e,u=u_e} \)

Thus, full linearized system is

\[ \begin{align*}
\dot{z} &= Az + Bu \\
w &= Cz + Du \\
&= y - y_e \\
v &= u - u_e
\end{align*} \]
Remarks:
1. Definition of linear input-output behavior:
   \[\begin{align*}
   &\text{if } x \to y \\
   &\text{then } ax \to ay \\
   &\text{if } x_1 \to y_1 \\
   &\text{and } x_2 \to y_2 \\
   &\text{then } x_1 + x_2 \to y_1 + y_2 \\
   \end{align*}\]

Example non-linear function:
\[\begin{align*}
   \text{if } x \to x^2 \\
   \text{then } ax \to ax^2 \\
   x_1 + x_2 \to x_1^2 + x_2^2 + x_1 x_2 \neq x_1^2 + x_2^2
   \end{align*}\]

Example linear function: \( y = ax \)
- the above conditions are satisfied

2. Solution to linear system (3) is convolution equation:
\[\begin{align*}
   y(t) &= Ce^{At} x(0) + \int_0^t Ce^{A(t-\tau)} B u(\tau) d\tau + Du(t)
   \end{align*}\]
if no input \((u = 0)\), solution is just \( y(t) = Ce^{At} x_0 \).

3. Stability of system with no input depends on the eigenvalues of the \( A \) matrix.
   - If \( A \) is diagonalizable with transform \( T \), \( AT = TD \), \( D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \)

   \[A = TD T^{-1}\]

   Then solution \( y = Ce^{At} x_0 \)
   \[= C \begin{bmatrix} 1 & A t & A^2 t^2 & \ldots \end{bmatrix} x_0\]
   \[= C \begin{bmatrix} T T^r & T A T^r & (T A T^r)^2 & \ldots \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix}\]
   \[= C T \begin{bmatrix} e^{At} & e^{A t^2} & \ldots \end{bmatrix} T^{-1} x_0 \]
   \[= C T \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \ldots \end{bmatrix} T^{-1} x_0 \]
   goes to zero if and only if all \( \Re \lambda_i < 0 \).
   - Can give similar proof using Jordan form of \( A \)
   - If \( A \) is not diagonalizable.
so, back to example, the system is stable if the eigenvalues of $A$ are in the left half of the complex plane.

First, let $z = x - te = [x - t] - \frac{t}{2} z$

find $A = \frac{df}{dx} = \begin{bmatrix} a & -a \\ b & b-c \frac{(d+zx)^2-2z}{(d+zx)^2} \end{bmatrix} = \begin{bmatrix} a & -a \\ -b & b-c \frac{d}{(d+zx)^2} \end{bmatrix} |_{te=(1,1)}$

$$= \begin{bmatrix} a & -a \\ -b & b-c \frac{d}{(d+1)^2} \end{bmatrix} = \begin{bmatrix} a & -a \\ -b & 0 \end{bmatrix}$$

and we know $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by inspection.
eigvals of \( A \): \( \det \begin{pmatrix} A - \lambda I \end{pmatrix} = 0 \Rightarrow (a-\lambda)(s) - (\xi)(s) = 0 \)
\[ s^2 - a \xi s - ab = 0 \]
\[ s = \frac{a \xi \pm \sqrt{a^2 \xi^2 + 4ab}}{2} \]
unstable for all \( ab > 0 \) \( \Rightarrow \) unstable.

Let's consider state feedback to stabilize.

Is it reachable? Reachability test: reachability matrix \([ B \ A B ]\)
full rank, can test with \( \det [ B \ A B ] \)
if not zero \( \Rightarrow \) full rank,
(matrix will be square for single input systems)

\[
[B \ A B] = \begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix}
\text{ test: } \det [0 \ 1 ; -a \ 0 ] = 0 + a \neq 0 \Rightarrow \text{ reachable.}
\]

to stabilize, since system is reachable, we can place eigenvalues

to be any value we want, if the eigenvalues of the linearized system are stable then the full nonlinear system is stable (in the neighborhood of that equilibrium point).

Consider state feedback of form \( u = -Kx = -k_1 x_1 - k_2 x_2 \)

\( K = \begin{bmatrix} k_1 \ k_2 \end{bmatrix} \)

\( z = x - x_e \), then linearized system is \( \dot{z} = Az + Bu \)

\( = Az - BKz \)

\( = (A - BK)z \)
suppose we want our eigenvalues $s$ to satisfy the characteristic equation $s^2 + 2bw_0 s + w_0^2 = 0$ ($\Rightarrow$ stable eigenvalues for $w_0$, $s > 0$)

find matrix $K$:

want $\det(A-BK)-sI) = 0$ to have same form.

\[
A-BK = \begin{bmatrix} a & -a \\ -b & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a-k_1 & -a-k_2 \\ -b & 0 \end{bmatrix}
\]

\[
\det(A-BK-sI) = \det \begin{bmatrix} a-s & -a \\ -b-k_1 & -k_2-s \end{bmatrix} = 0
\]

\[
= (a-s)(-k_2-s) - (a)(-b-k_1)
\]

\[
= s^2 - as + k_2 s - ak_2 - ab - ak_1
\]

\[
= s^2 + (-a+ak_2)s + (-ak_2 - ab - ak_1) = 0
\]

(1) $-a + k_2 = 2b w_0$

(2) $-ak_2 - ab - ak_1 = w_0^2$

solve:

(1) $k_2 = 2bw_0 - a$

(2) $-2abw_0 + a^2 c - ab - ak_1 = w_0^2$

$-ak_1 = w_0^2 + ab + a^2 + 2abw_0$

$k_1 = \frac{w_0^2 - b - a - 2bw_0}{a}$

choose $w_0 = 1$

$s_2 = 0.15$

so $u = k_1 z_1 - k_2 z_2$

$= -k_1(x-1) - k_2(x_2-2)$
also, we could have used matlab function: \( k = \text{place}(A, B, [s, s_2, \ldots, s_n]) \)

show stability with a lyapunov function at the

\( V = z^T P z \) 
\( V \) is scalar, to prove stability
need \( V > 0 \) for \( z \neq 0 \), \( V = 0 \) at \( z = 0 \), (pos det)
and \( V \leq 0 \) for \( z \neq 0 \) (neg det.)

then \( \dot{V} = \frac{dV}{dt} = z^T P z + z^T P z \)

let \( \hat{A} \) be matrix with state feed back \( \hat{A} = A - BK \)
and nonlinear system is \( \dot{z} = \hat{A} z + \tilde{F}(z) \) where \( \tilde{F} \) is higher order terms
\( \tilde{F}(z) = F(z + \epsilon) - A z \)

then \( \dot{V} = (\hat{A} z + \tilde{F}(z))^T P z + z^T P (\hat{A} z + \tilde{F}(z)) z \)
\( = z^T (\hat{A}^T P + P \hat{A}) z + \text{higher order } O(\epsilon) \)
\( = z^T Q z \)
need only \( Q > 0 \) for \( \dot{V} < 0 \)
choose \( Q = I \) => solve for \( P \).

\( \Rightarrow \) will find one if \( A \) stable (which it is here)
\( P = \text{lyap}(A, Q) \) in MATALB

proves stability of nonlinear system because \( O(\epsilon) \) terms
will always be smaller than \( O(\epsilon) \) terms in any sufficiently small
neighborhood.
\[ x' = a (x - y) \]
\[ y' = b (y - x) - b (d + 1)^2 / d y (d + y) + e \]

\[ a = 1.5/48 \]
\[ b = 1.5/69 \]
\[ e = 16.5/69 \]
\[ d = .1 \]

The backward orbit from (0.82, 0.8) \(\rightarrow\) a nearly closed orbit.
Ready.
The forward orbit from (1.1, 1) left the computation window.
The backward orbit from (1.1, 1) left the computation window.
Ready.
The backward orbit from (0.9, 0.067) \rightarrow a nearly closed orbit.
Ready.
The forward orbit from (0.79, 0.45) \rightarrow a possible eq. pt. near (1, 1).
The backward orbit from (0.79, 0.45) left the computation window.
Ready.