CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

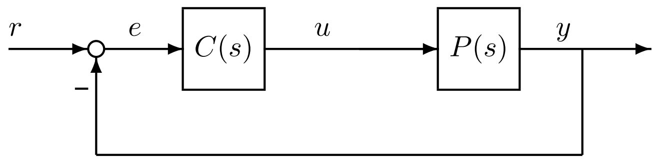
CDS 131

R. Murray — **Homework Set #1** — Issued: 30 Sep 2020
Fall 2020 — Due: 7 Oct 2020

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Complete 5 of the following 6 problems:

1. [P. Seiler, U. Minnesota] Consider the following feedback diagram [10 pts]



and let $L(s) := P(s)C(s)$ denote the loop transfer function for the feedback system shown below. Consider two pairs of plants and controllers:

i) $C(s) = \frac{10(s+3)}{s}$ and $P(s) = \frac{-0.5(s^2-2500)}{(s-3)(s^2+50s+1000)}$
ii) $C(s) = \frac{0.4s+1}{s}$ and $P(s) = \frac{1}{s+1}$

Perform the following calculations for each plant/controller pair:

(a) Verify that the feedback system is stable.
(b) Use the Bode plot of $L(s)$ to find all phase-crossover frequencies, i.e. frequencies ω_0 such that $\angle L(j\omega_0) = -180^o$. Use this information to compute the gain margin(s) of the feedback system.
(c) For each gain margin g_0 calculated in part b), construct the closed-loop using the perturbed loop transfer function $g_0 L(s)$. Verify that the closed-loop has poles at $\pm j\omega_0$ and hence the gain variation g_0 causes instability.
(d) Use the Bode plot of $L(s)$ to find all gain-crossover frequencies, i.e. frequencies ω_0 such that $|L(j\omega_0)| = 1$. Use this information to compute the phase and time delay margins of the feedback system.

You can use MATLAB, Python, Mathematica or other computer programs to compute your answers, but make sure to include any code that you use and also that you clearly label all important features in your answers.

2. [FBS2e 2.4] Consider a closed loop system with process dynamics and a PI controller modeled by [10 pts]

$$\frac{dy}{dt} + ay = bu, \qquad u = k_\text{p}(r-y) + k_\text{i}\int_0^t \big(r(\tau) - y(\tau)\big)\, d\tau,$$

where r is the reference, u is the control variable, and y is the process output.

(a) Derive a differential equation relating the output y to the reference r by direct manipulation of the equations and compute the transfer function $H_{yr}(s)$.

(b) Draw a block diagram of the system and derive the transfer functions of the process $P(s)$ and the controller $C(s)$.

(c) Use block diagram algebra to compute the transfer function from reference r to output y of the closed loop system and verify that your answer matches your answer in part (a).

3. [FBS2e 9.2] Let $G(s)$ be the transfer function for a linear system. Show that if we apply an input $u(t) = A\sin(\omega t)$, then the steady-state output is given by $y(t) = |G(i\omega)|A\sin(\omega t + \arg G(i\omega))$. (Hint: Start by showing that the real part of a complex number is a linear operation and then use this fact.) [10 pts]

4. [FBS2e 9.7] Consider the linear state space system [10 pts]

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

(a) Show that the transfer function is

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

where the coefficients for the numerator polynomial are linear combinations of the *Markov parameters* CA^iB, $i = 0, \ldots, n-1$:

$$b_1 = CB, \quad b_2 = CAB + a_1 CB, \quad \ldots, \quad b_n = CA^{n-1}B + a_1 CA^{n-2}B + \cdots + a_{n-1}CB$$

and $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.

(b) Compute the transfer function for a linear system in reachable canonical form and show that it matches the transfer function given above.

5. [FBS2e 9.13] Consider the differential equation [10 pts]

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + b_2 \frac{d^{n-2}u}{dt^{n-2}} + \cdots + b_n u.$$

(a) Let λ be a root of the characteristic equation

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0.$$

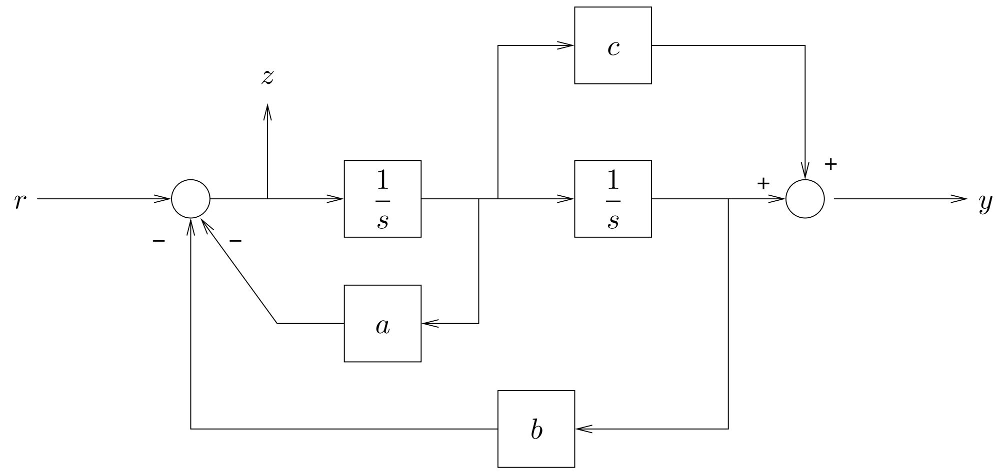
Show that if $u(t) = 0$, the differential equation has the solution $y(t) = e^{\lambda t}$.

(b) Let κ be a zero of the polynomial

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n.$$

Show that if the input is $u(t) = e^{\kappa t}$, then there is a solution to the differential equation that is identically zero.

6. Consider the block diagram for the following second-order system [10 pts]



(a) Compute the transfer function H_{yr} between the input r and the output y.

(b) Show that the following state space system has the same transfer function, with the appropriate choice of parameters:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$y = \begin{pmatrix} b_2 & b_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + dr$$

Give the values of a_i, b_i, and d that correspond to the transfer function you computed in (a).

(c) Compute the transfer function H_{zr} between the input r and the output z. (Hint: It is *not* $H_{zr} = 1$.)