CDS 101: Lecture 8.1
Frequency Domain Design using PID

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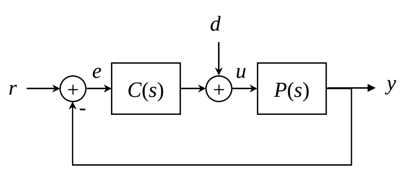
Goals:

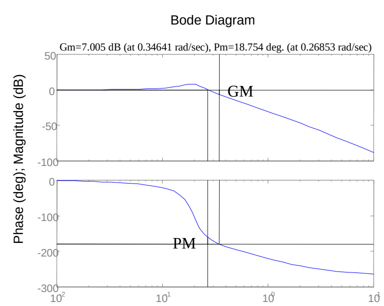
- Describe the use of frequency domain performance specifications
- Show how to use "loop shaping" using PID to achieve a performance specification

Reading:

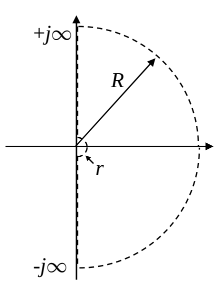
- Åström and Murray, *Analysis and Design of Feedback Systems,* 7.6ff and Ch 8

Review from Last Week





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



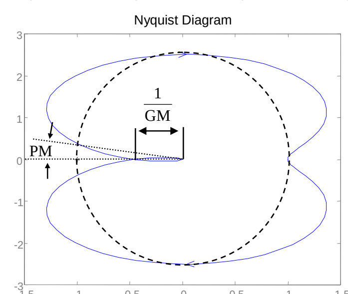
Thm (Nyquist).

P # RHP poles of $L(s)$

N # CW encirclements

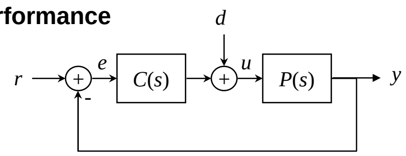
Z # RHP zeros

$$Z = N + P$$



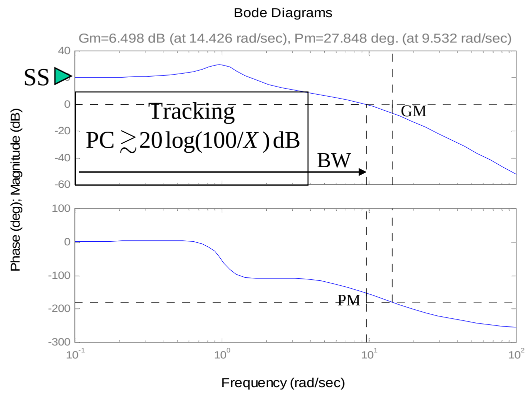
Frequency Domain Performance Specifications

Specify bounds on the loop transfer function to guarantee desired performance



$$L(s) = P(s)C(s)$$

$$H_{er} = \frac{1}{1+L} \qquad H_{yr} = \frac{L}{1+L}$$

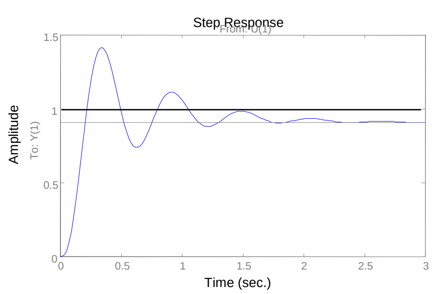


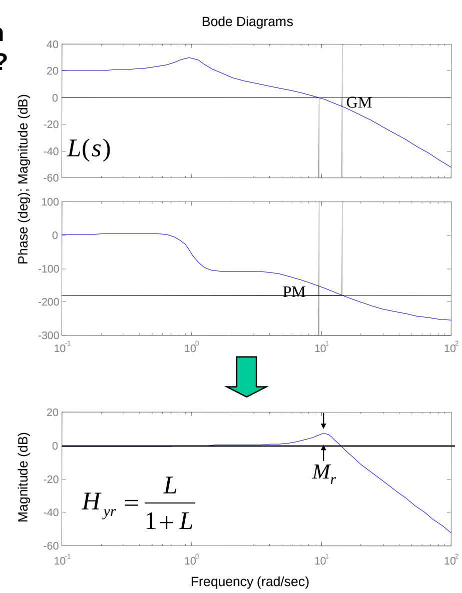
- Steady state error:
  $H_{er}(0) = 1/(1+L(0)) \approx 1/L(0)$
  $\Rightarrow$ zero frequency ("DC") gain ▷
- Bandwidth: assuming ~90˚ phase margin
  $$\frac{L}{1+L}(j\omega_c) \approx \left|\frac{1}{1+j}\right| = \frac{1}{\sqrt{2}}$$
  $\Rightarrow$ sets crossover freq →
- Tracking: X% error up to frequency $\omega_t \Rightarrow$ determines gain bound (1 + PC > 100/X) ☐

Relative Stability

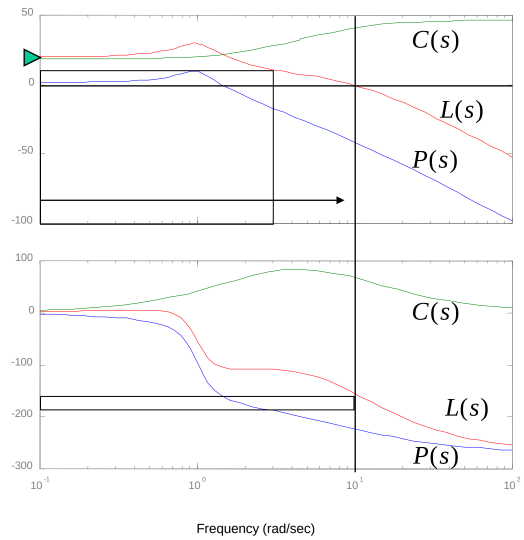
Relative stability: how stable is system to disturbances at certain frequencies?

- System can be stable but still have bad response at certain frequencies
- Typically occurs if system has low phase margin $\Rightarrow$ get resonant peak in closed loop (M_r) + poor step response
- Solution: specify minimum phase margin. Typically 45˚ or more





Overview of Loop Shaping



Performance specification

- Steady state error
- Tracking error
- Bandwidth
- Relative stability

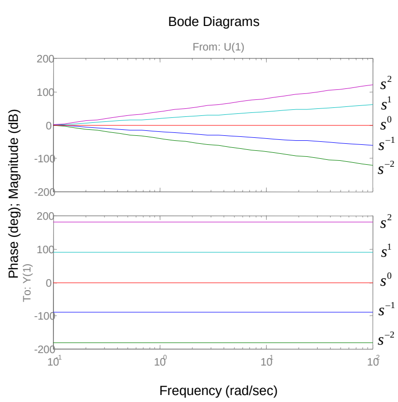
Approach: "shape" loop transfer function using $C(s)$

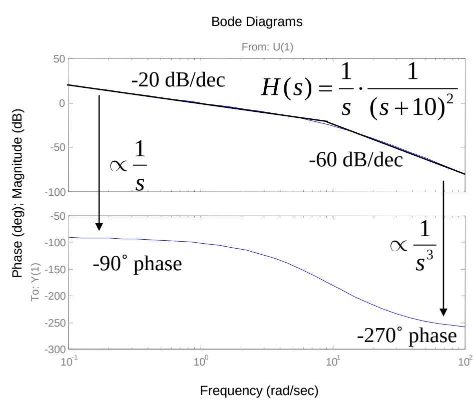
- $P(s)$ + specifications given
- $L(s) = P(s)\, C(s)$
  - Use $C(s)$ to choose desired shape for $L(s)$
- Important: can't set gain and phase independently

Gain/phase relationships

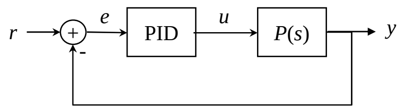
Gain and phase for transfer function w/ real coeffs are not independent

- Given a given shape for the gain, there is a unique "minimum phase" transfer function that achieves that gain at the specified frequencies
- Basic idea: slope of the gain determines the phase
- Implication: you have to tradeoff gain versus phase in control design





Overview: PID control



$$u = K_p e + K_I \int e + K_D \dot{e}$$

Intuition

- Proportional term: provides inputs that correct for “current” errors
- Integral term: insures *steady state* error goes to zero
- Derivative term: provides “anticipation” of upcoming changes

A bit of history on “three term control”

- First appeared in 1922 paper by Minorsky: “Directional stability of automatically steered bodies” under the name “three term control”
- Also realized that “small deviations” (linearization) could be used to understand the (nonlinear) system dynamics under control

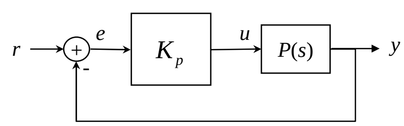
Utility of PID

- PID control is most common feedback structure in engineering systems
- For many systems, only need PI or PD (special case)
- Many tools for tuning PID loops and designing gains (see reading)

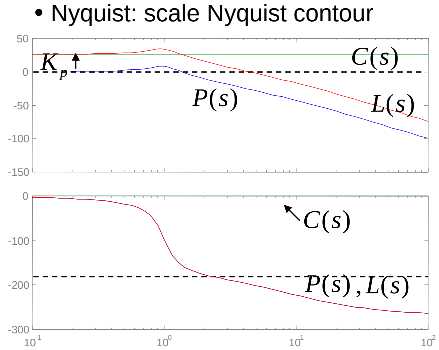
Proportional Feedback

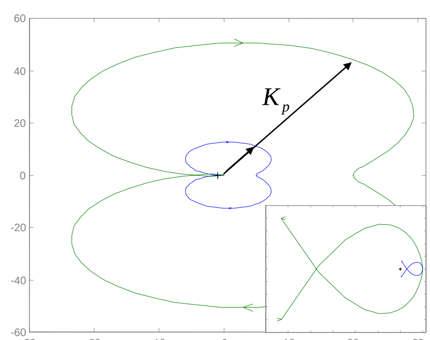
Simplest controller choice: $u = K_p e$

- Effect: lifts gain with no change in phase
- Good for plants with low phase up to desired bandwidth
- Bode: shift gain up by factor of K_p
- Nyquist: scale Nyquist contour



$K_p > 0$

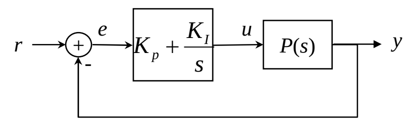




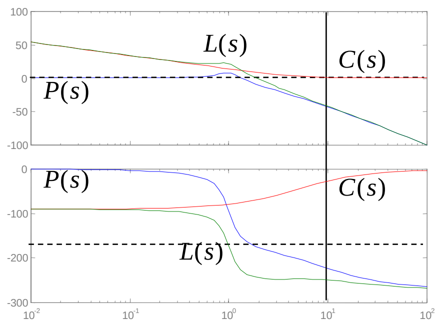
Proportional + Integral Compensation

Use to eliminate steady state error

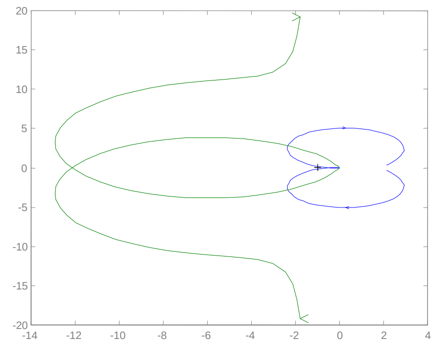
- Effect: lifts gain at low frequency
- Gives *zero* steady state error
- Bode: infinite SS gain + phase lag
- Nyquist: no easy interpretation
- Note: this example is *unstable*



$K_p > 0 \quad K_I > 0$



$\omega_z = K_I / K_p$



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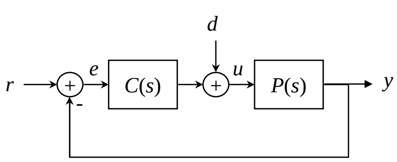
Proportional + Integral + Derivative (PID)

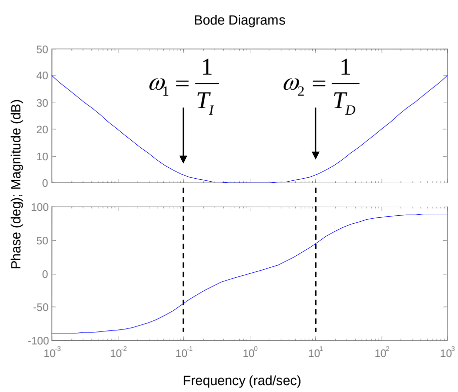


$$C(s) = K_p + K_I \cdot \frac{1}{s} + K_D s$$
$$= k(1 + \frac{1}{T_I s} + T_D s)$$
$$= \frac{kT_D}{T_I} \frac{(s + 1/T_I)(s + 1/T_D)}{s}$$

Transfer function for PID controller

$$u = K_p e + K_I \int e + K_D \dot{e}$$
$$\downarrow$$
$$H_{ue}(s) = K_p + K_I \cdot \frac{1}{s} + K_D s$$

- Idea: gives high gain at low frequency plus phase lead at high frequency
- Place ω_1 and ω_2 below desired crossover freq



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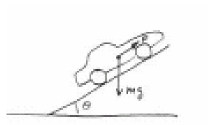
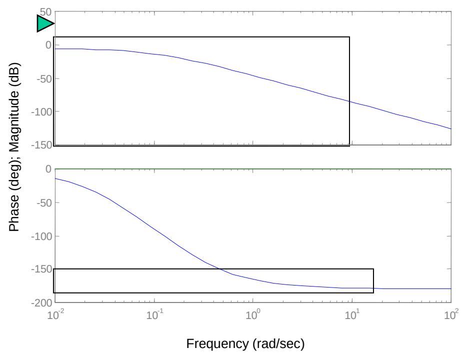
Example: Cruise Control using PID - Specification



$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}$$



Performance Specification

- ≤ 1% steady state error
  - Zero frequency gain > 100
- ≤ 10% tracking error up to 10 rad/sec
  - Gain > 10 from 0-10 rad/sec
- ≥ 45° phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

Observations

- Purely proportional gain won't work: to get gain above desired level will not leave adequate phase margine
- Need to increase the phase from ~0.5 to 2 rad/sec and increase gain as well

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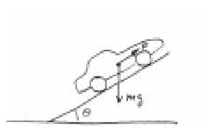
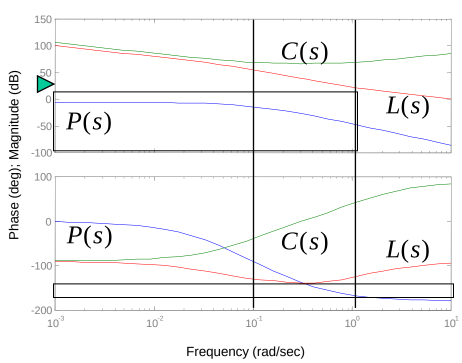
Example: Cruise Control using PID - Design



$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}$$



Approach

- Use integral gain to make steady state error small (zero, in fact)
- Use derivative action to increase phase lead in the cross over region
- Use proportional gain to give desired bandwidth

Controller

$$C(s) = 2000\frac{s^2 + 1.1s + 0.1}{s}$$
$$= 2200 + \frac{200}{s} + 2000s$$

Closed loop system

- Very high steady state gain
- Adequate tracking @ 1 rad/sec
- ~80° phase margin

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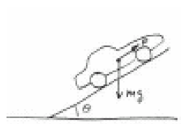
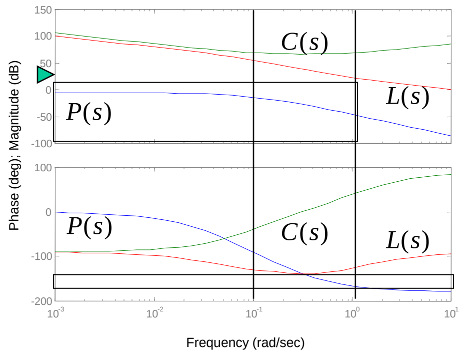
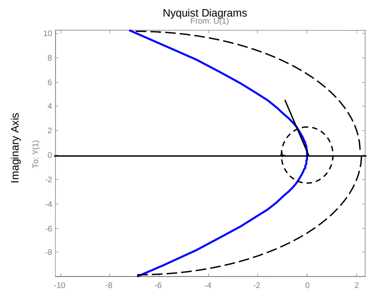
Example: Cruise Control using PID - Verification

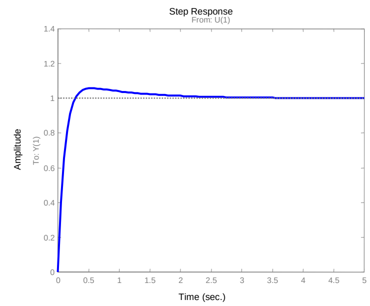


$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}$$

$$C(s) = 2000\frac{s^2 + 1.1s + 0.1}{s}$$







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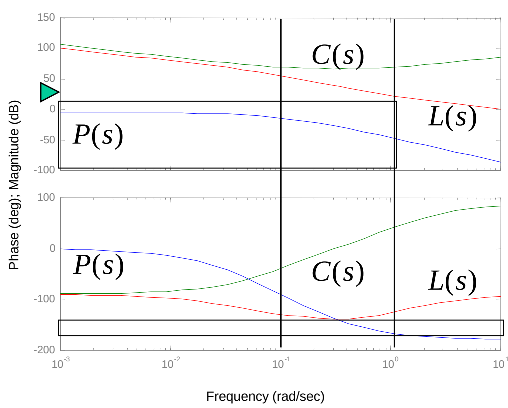
Summary: Frequency Domain Design using PID

Loop Shaping for Stability & Performance

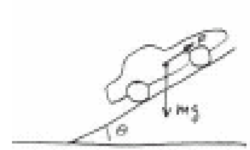
- Steady state error, bandwidth, tracking

$$H_{ue}(s) = K_p + K_I \cdot \frac{1}{s} + K_D s$$



Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, PI, PID



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