CDS 101: Lecture 7.1
Loop Analysis of Feedback Systems

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8 November 2004

Goals:
- Show how to compute closed loop stability from open loop properties
- Describe the Nyquist stability criterion for stability of feedback systems
- Define gain and phase margin and determine it from Nyquist and Bode plots
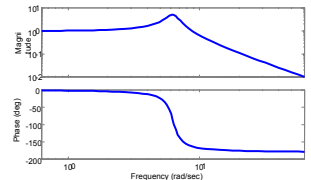
Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 7
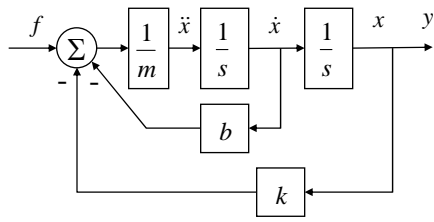

Review from Last Week

$$u = A\sin(\omega t)$$

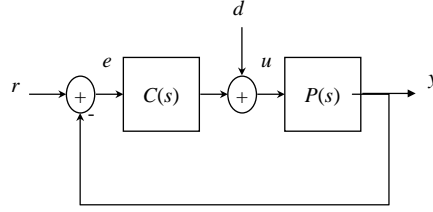$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

$$y_{ss} = |H(j\omega)|A \cdot \sin(\omega t + \angle H(j\omega))$$

$$H(s) = C(sI - A)^{-1}B + D$$

$$H_{y_2 u_1} = H_{y_2 u_2} H_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$

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Closed Loop Stability

Q: how do open loop dynamics affect the closed loop stability?

- Given open loop transfer function $C(s)P(s)$ determine when system is stable

Brute force answer: compute poles closed loop transfer function

$$H_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of H_{yr} = zeros of $1 + PC$
- Easy to compute, but not so good for design

Alternative: look for conditions on *PC* that lead to instability

- Example: if $PC(s) = $ -1 for some $s = j\omega$, then system is *not* asymptotically stable
- Condition on *PC* is much nicer because we can *design PC*(*s*) by choice of *C*(*s*)
- However, checking *PC*(*s*) = -1 is not enough; need more sophisticated check

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Game Plan: Frequency Domain Design

Goal: figure out how to *design C*(*s*) so that 1+*C*(*s*)*P*(*s*) is stable *and* we get good performance
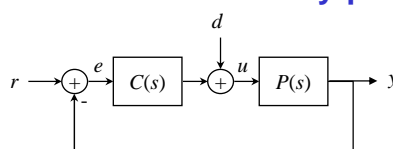
$$H_{yr} = \frac{PC}{1+PC}$$

- Poles of H_{yr} = zeros of $1 + PC$
- Would also like to "shape" H_{yr} to specify performance at different frequencies

- Low frequency range:

$$PC \gg 1 \quad \Rightarrow \quad \frac{PC}{1+PC} \approx 1$$

(good tracking)

- Bandwidth: frequency at which closed loop gain = ½ $\Rightarrow$ open loop gain ≈ 1
- Idea: use *C*(*s*) to *shape PC* (under certain constraints)
- Need tools to analyze stability and performance for closed loop given *PC*

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Nyquist Criterion

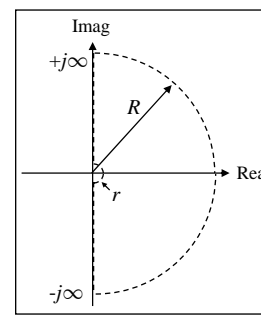**Determine stability from (open) loop transfer function, $L(s) = P(s)C(s)$.**

• Use "principle of the argument" from complex variable theory (see reading)

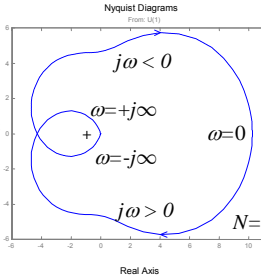**Thm (Nyquist).** Consider the Nyquist plot for loop transfer function $L(s)$. Let

P # RHP poles of $L(s)$

N # clockwise encirclements of -1

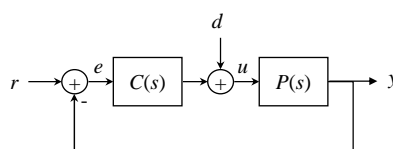Z # RHP zeros of $1 + L(s)$

Then

$$Z = N + P$$

• Nyquist "D" contour
• Take limit as $r \to 0$, $R \to \infty$
• Trace from $-\infty$ to $+\infty$ along imaginary axis

• Trace frequency response for $L(s)$ along the Nyquist "D" contour
• Count net # of clockwise encirclements of the -1 point

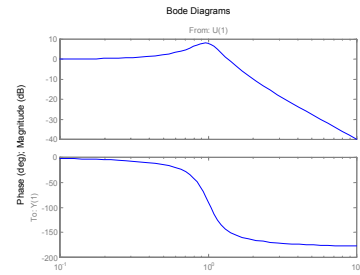8 Nov 04   R. M. Murray, Caltech CDS   5
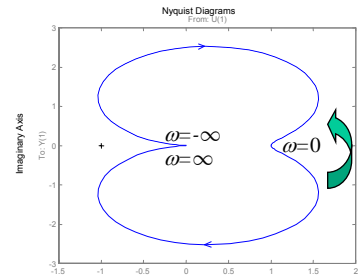
Simple Interpretation of Nyquist

Basic idea: avoid positive feedback
• If $L(s)$ has 180° phase (or greater) and gain greater than 1, then signals are amplified around loop
• Use when phase is monotonic
• General case requires Nyquist

Can generate Nyquist plot from Bode plot + reflection around real axis
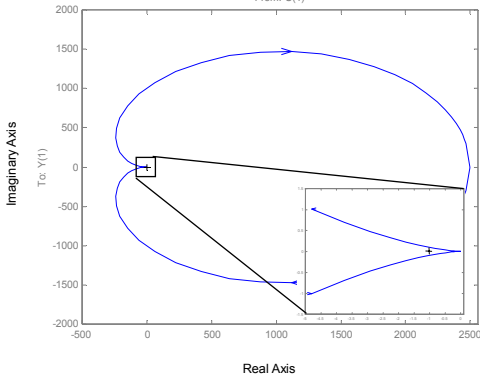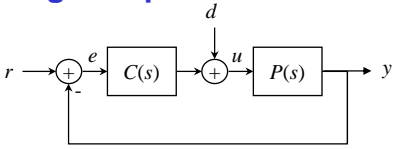
bode(sys)   nyquist(sys)

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Example: Proportional + Integral* speed controller

d

$r \rightarrow \oplus \overset{e}{\rightarrow} \boxed{C(s)} \rightarrow \oplus \overset{u}{\rightarrow} \boxed{P(s)} \rightarrow y$

Nyquist Diagrams
From: U(1)

$$P(s) = \frac{1/m}{s+b/m} \cdot \frac{r}{s+a}$$

$$C(s) = K_p + \frac{K_i}{s+0.01}$$

Remarks
- $N = 0$, $P = 0 \Rightarrow Z = 0$ (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don't have to compute closed loop response

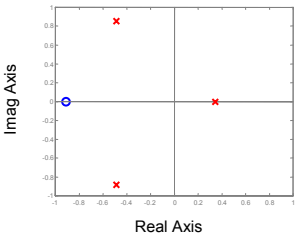* slightly modified; more on the design of this compensator in next week's lecture

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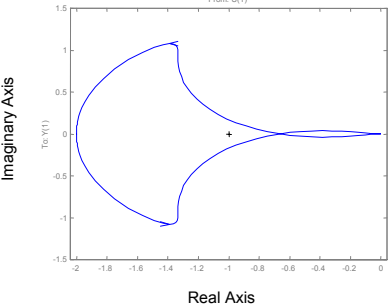

More complicated systems

What happens when open loop plant has RHP poles?
- 1 + PC has singularities inside D countour $\Rightarrow$ these must be taken into account

Pole-zero map

Nyquist Diagrams
From: U(1)

$$L(s) = \frac{s+1}{s-0.5} \cdot \frac{1}{s^2+s+1}$$

unstable pole

$$N = -1, P = 1 \Rightarrow Z = N+P = 0 \text{ (stable)}$$

$$\frac{1}{1+L} = \frac{s+1}{(s+0.35)(s+0.07+1.2j)(s+0.07-1.2j)} \checkmark$$

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Comments and cautions

Why is the Nyquist plot *useful*?

- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives *insight* into stability and robustness; very useful for reasoning about stability

Nyquist plots for systems with poles on the *jω* axis

$$H(s) = \frac{1}{s(s+1)^2}$$

- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\epsilon+0j)$ to determine direction

$\omega=0^-$

$\omega=+j\infty$

$\omega=-j\infty$

$\omega=0^+$

Cautions with using MATLAB

- MATLAB doesn't generate portion of plot for poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)

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Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not *how* stable

Gain margin

- How much we can modify the *loop gain* and still have the system be stable
- Determined by the location where the loop transfer function crosses 180° phase

Phase margin

- How much we can add "phase delay" and still have the system be stable
- Determined by the phase at which the loop transfer function has unity gain
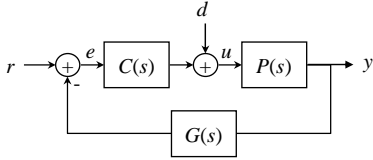
Bode plot interpretation

- Look for gain = 1, 180° phase crossings
- MATLAB: margin(sys)

Nyquist Diagram

$\frac{1}{GM}$

PM

Bode Diagram

Gm=7.005 dB (at 0.34641 rad/sec), Pm=18.754 deg. (at 0.26853 rad/sec)

GM

Phase (deg); Magnitude (dB)

PM

Fre

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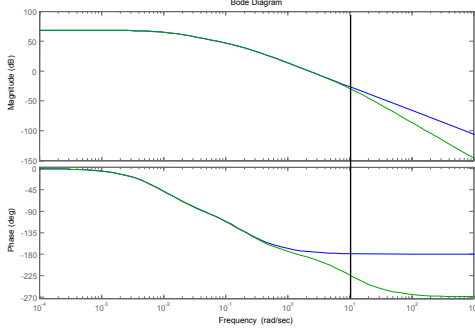
Example: cruise control

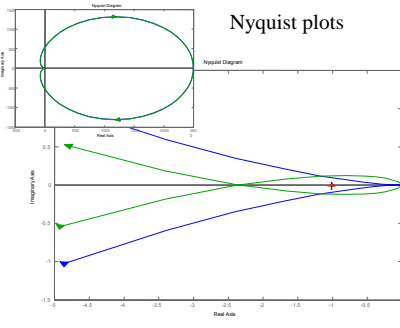$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s + 10}$$

Effect of additional sensor dynamics

• New speedometer has pole at $s = 10$ (very fast); problems develop in the field
• What's the problem? A: insufficient phase margin in original design (not robust)
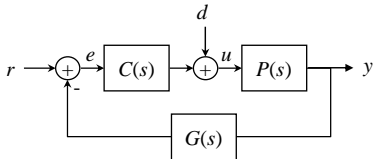
Nyquist plots

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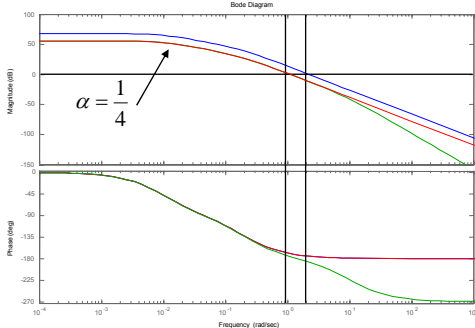## Preview: control *design*

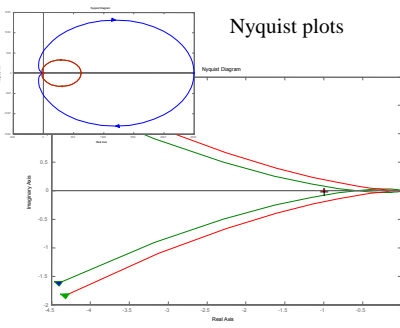$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}$$

$$C(s) = \alpha \left(K_p + \frac{K_i}{s + 0.01} \right)$$

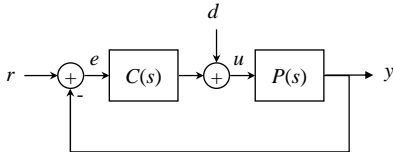$$G(s) = \frac{10}{s + 10}$$

Approach: Increase phase margin

• Increase phase margin by reducing gain $\Rightarrow$ can accommodate new sensor dynamics
• Tradeoff: lower gain at low frequencies $\Rightarrow$ less bandwidth, larger steady state error

$\alpha = \frac{1}{4}$

Nyquist plots
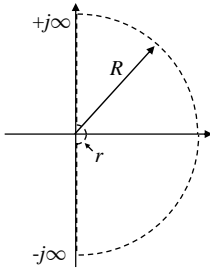
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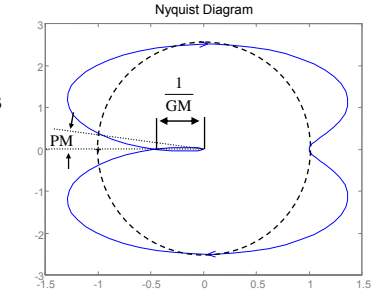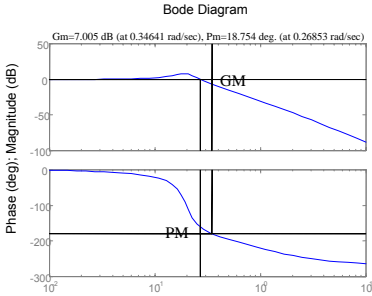# Summary: Loop Analysis of Feedback Systems

- Nyquist criteria for loop stability
- Gain, phase margin for robustness

Thm (Nyquist).

P # RHP poles of $L(s)$

N # CW encirclements

Z # RHP zeros

$$Z = N + P$$

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