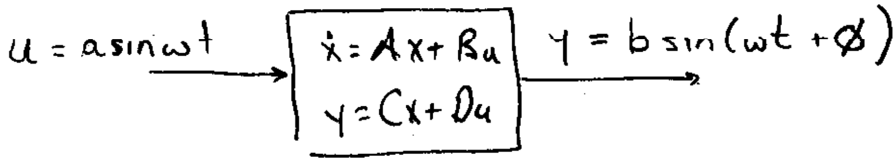
RMM 3 Nov 04

Exponential response



To compute y given u, we look at the exponential response

$$u = e^{st} = e^{\sigma t}(\cos\omega t + i\sin\omega t) \qquad s = \sigma + i\omega$$

Remark: This is a complex signal, but is really a shortcut for computing $\sin(\omega t + \phi) = \alpha\cos\omega t + \beta\sin\omega t$

Compute y using the convolution integral

$$\begin{aligned}
y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du \\
&= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Be^{s\tau}d\tau + Du \\
&= Ce^{At}x(0) + Ce^{At}\int_0^t e^{sI\tau}e^{-A\tau}B\,d\tau + Du \\
&= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}B\,d\tau + Du \\
&= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}e^{(sI-A)\tau}B\Big|_0^t + Du \\
&= Ce^{At}x(0) - Ce^{At}(sI-A)^{-1}B + Ce^{At}(sI-A)^{-1}e^{(sI-A)t}B + Du \\
&= \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{Transient} \to 0 \text{ if } A \text{ stable}} + \underbrace{\left[C(sI-A)^{-1}B + D\right]e^{st}}_{\text{Steady state}}
\end{aligned}$$

$y_{ss} = H(s)e^{st}$

Second order systems

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = u(t)$$

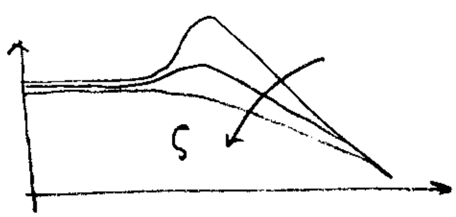
Recall from lecture 3.2

- System is asymptotically stable if $\zeta > 0$
- Particular solution is given by

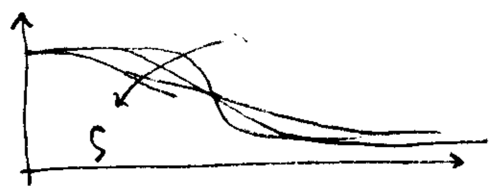
$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Recall from homework #3



Complicated formulas for magnitude & phase



Let's recreate this plot using <u>transfer functions</u>

State space representation

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

RMM 2 Nov 02

②

$$H(s) = C(sI - A)^{-1}B + \cancel{D}^{\,=0}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} = \begin{bmatrix} s & -1 \\ \omega_n^2 & s+2\zeta\omega_n \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+2\zeta\omega_n) + \omega_n^2} \begin{bmatrix} s+2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix}$$

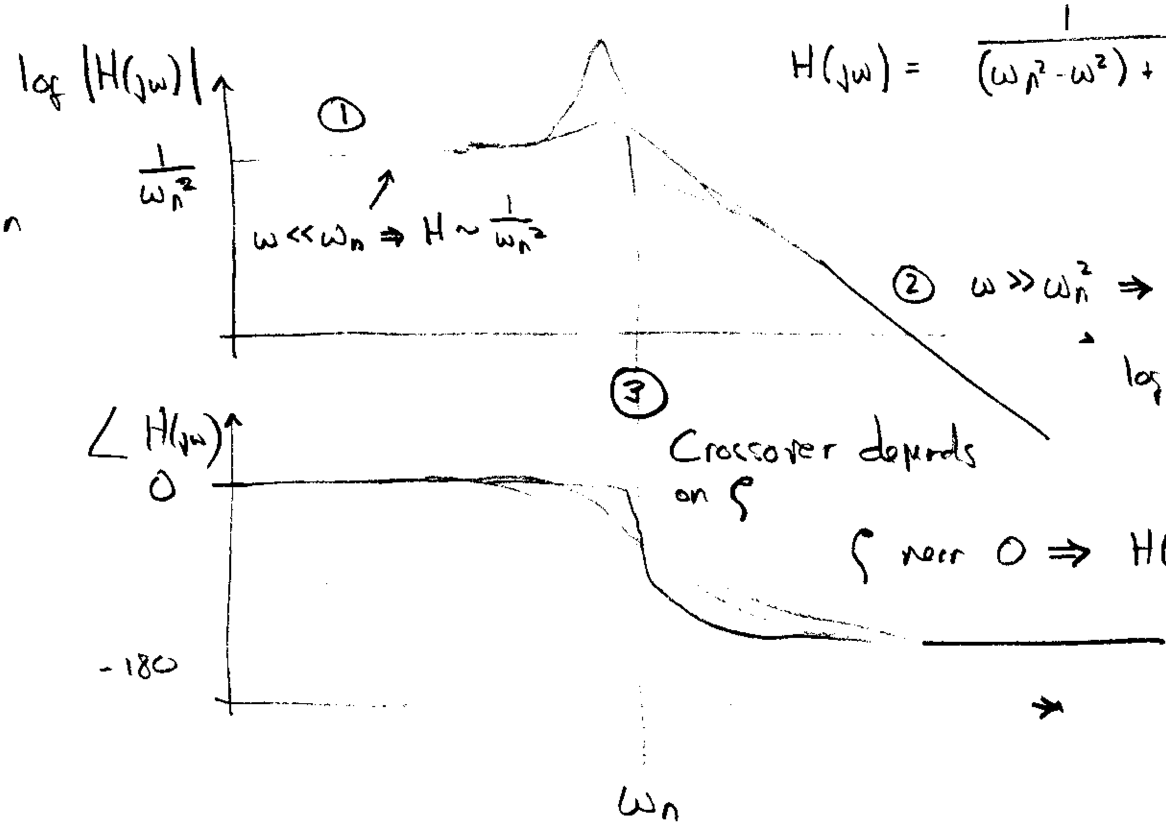
$$= \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \begin{bmatrix} & '' & \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & 1 \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{H(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Now we can use $H(j\omega)$ to get Bode plot

$$H(j\omega) = \frac{1}{(\omega_n^2 - \omega^2) + 2\zeta\omega_n\omega j}$$



Doug: fill in ① then ② then ③

② $\omega \gg \omega_n^2 \Rightarrow H \sim \frac{-1}{\omega^2}$

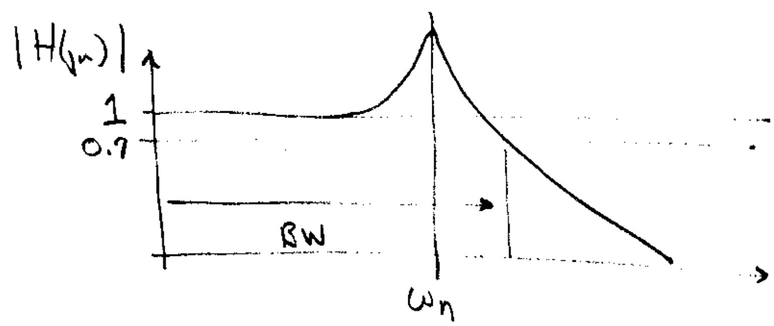
$\log|H| = -2\log\omega$

ζ near $0 \Rightarrow H(j\omega) \to \infty$

Remarks

1. Note that use of transfer function to compute frequency response is *much* easier than solving ODEs

2. Second order systems lead to definition of bandwidth, resonant frequency & "Q"



$H = \dfrac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

BW = frequency at which mag drops $\sqrt{2}$ below unity

Resonant frequency = $\sqrt{1-\zeta^2}\,\omega_n$

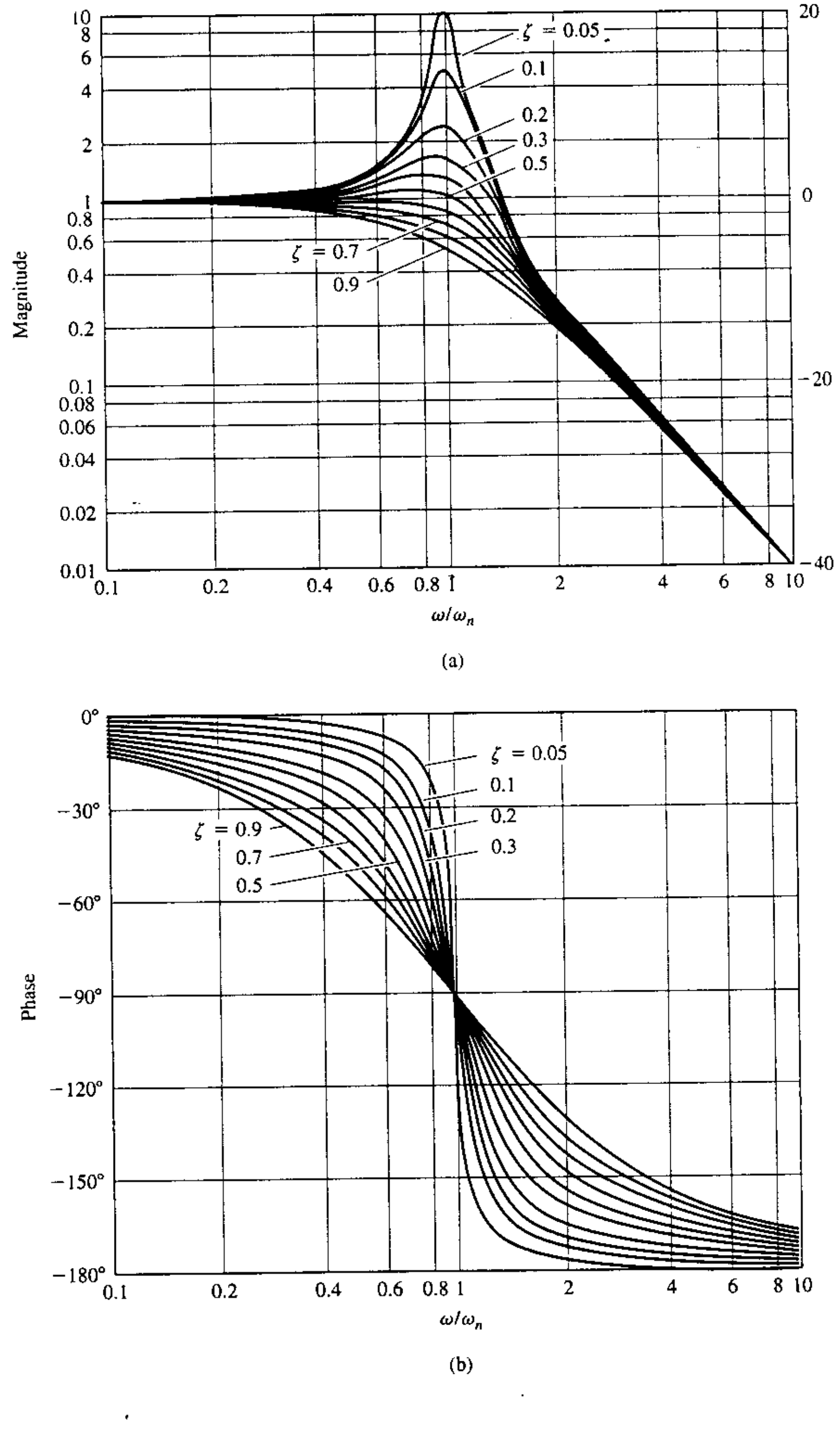
Quality factor, Q =

- used for systems that are *supposed* to resonate (lasers, harmonic oscillators, etc)

Details (if needed)

(3b)

FIGURE 6.2
(a) Magnitude and (b) phase of Eq. (6.7)



Sketching Bode plots

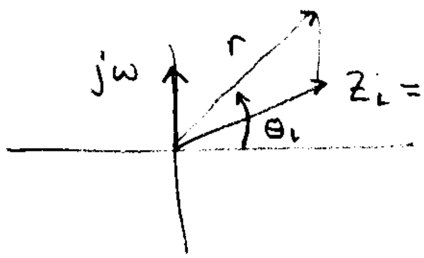
Given a transfer function $H(s) = \frac{n(s)}{d(s)}$, we can sketch Bode plot by looking at poles and zeros

$$H(s) = K \frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} \qquad z_i, p_i \in \mathbb{C}$$

$$H(j\omega) = K \frac{(j\omega+z_1)\cdots(j\omega+z_m)}{(j\omega+p_1)\cdots(s+p_n)}$$

Each term can be written as $r_i e^{j\theta_i}$ where

r_i = magnitude, θ_i = phase



$$r = \sqrt{\alpha^2+\beta^2}$$
$$\sigma = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$H(j\omega) = K \frac{r_1 e^{j\theta_1} \cdots r_m e^{j\theta_m}}{r_{m+1} e^{j\theta_{m+1}} \cdots r_{m+n} e^{j\theta_{m+n}}}$$

$$= K\, r_1 e^{j\theta_1} \cdots r_m e^{j\theta_m}\, r_{m+1} e^{-j\theta_{m+1}} \cdots r_{m+n} e^{-j\theta_{m+n}}$$

$$= \underbrace{K r_1 r_2 \cdots r_{m+n}}_{\text{mag}}\, e^{j\overbrace{}^{}\underbrace{(\theta_1 + \cdots + \theta_m - \theta_{m+1} - \cdots - \theta_{m+n})}_{\text{phase}}}$$

Bode plot

- log amplitude $\Rightarrow$ <u>add</u> $K, r_1, \ldots, r_{m+n}$
- linear phase $\Rightarrow$ <u>add</u> phase of zeros, <u>subtract</u> phase of poles

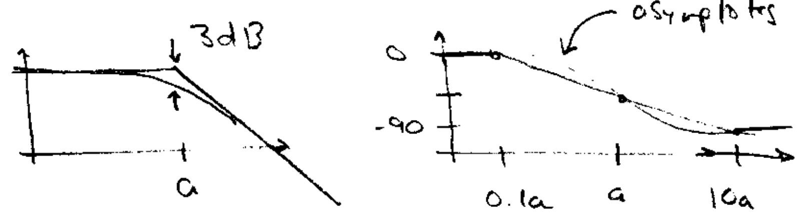
<u>Example</u> $H(s) = \frac{2(s+10)}{(s+1)(s+100)}$

(work out by plotting components & adding)
- talk about ignoring components that are far away

Details omitted

Bode plots for standard transfer functions

1. $H(s) = \frac{1}{s+a}$



2. $H(s) = s+b$

3. $H(s) = \frac{1}{s^2}$ etc

4. $H(s) = \frac{1}{s^2+bs+k}$
(complex pole)

<u>Example</u> Two coupled masses (from lecture)

Details on asymptotes (if needed)

6b

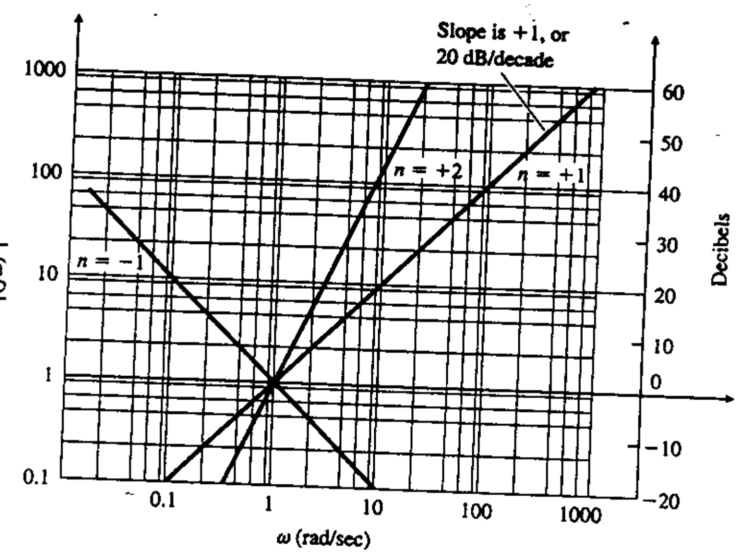


FIGURE 6.5
Magnitude of $(j\omega)^n$

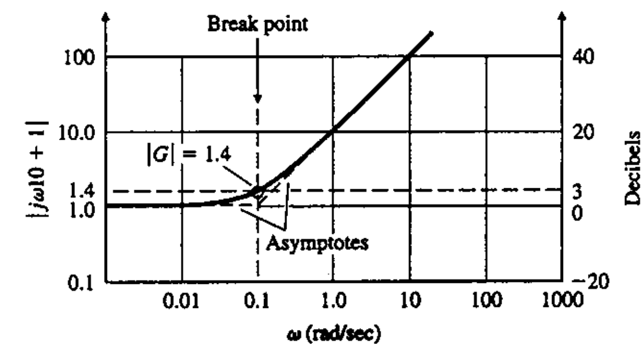


FIGURE 6.6
Magnitude plot for $j\omega\tau + 1|;\ \tau = 0.1$

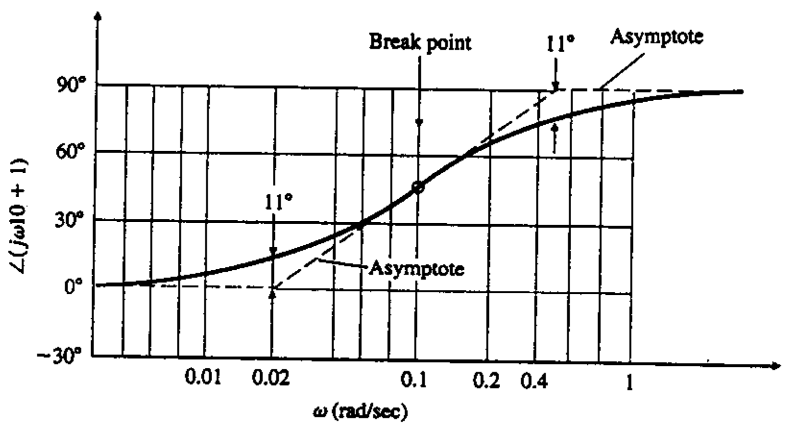


FIGURE 6.7
Phase plot for $j\omega\tau + 1|;\ \tau = 0.1$

Details on composition, if needed

5c

FIGURE 6.8
Composite plots:
(a) magnitude;
(b) phase;
(c) approximate phase