CDS 101: Lecture 4.1
Linear Systems

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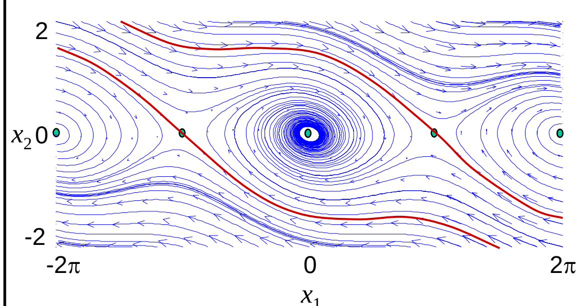
Goals:

- Describe linear system models: properties, examples, and tools
- Characterize stability and performance of linear systems in terms of eigenvalues
- Compute linearization of a nonlinear systems around an equilibrium point

Reading:

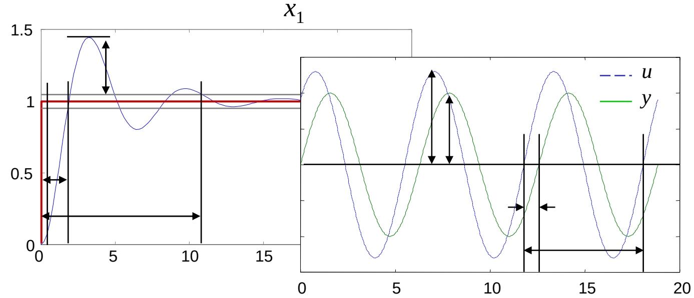
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 4
- Packard, Poola and Horowitz, *Dynamic Systems and Feedback,* Sections 19, 20, 22 (available via course web page)

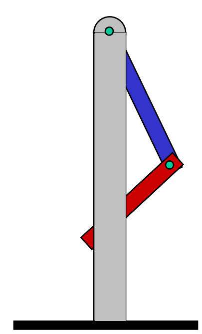
Review from Last Week



Key topics for this lecture

- Stability of equilibrium points
- Local versus global behavior
- Performance specification via step and frequency response





What is a *Linear* System?

Linearity of functions: $f : \mathbb{R}^n \to \mathbb{R}^m$

- Zero at the origin: $f(0) = 0$
- Addition: $f(x + y) = f(x) + f(y)$
- Scaling: $f(\alpha x) = \alpha f(x)$

$\Bigg\}$ $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

Canonical example:

$$f(x) = Ax$$

Linearity of *systems*: sums of *solutions*

Dynamical system

$$\dot{x} = Ax$$

$x(0) = x_{10} \quad \to x(t) = x_1(t)$

$x(0) = x_{20} \quad \to x(t) = x_2(t)$

$$\Downarrow$$

$$x(0) = \alpha x_{10} + \beta x_{20} \quad \to x(t) = \alpha x_1(t) + \beta x_2(t)$$

Control system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$x(0) = 0,\ u(t) = u_1(t) \quad \to y(t) = y_1(t)$

$x(0) = 0,\ u(t) = u_2(t) \quad \to y(t) = y_2(t)$

$$\Downarrow$$

$$x(0) = 0,\ u(t) = \alpha u_1(t) + \beta u_2(t) \quad \to y(t) = \alpha y_1(t) + \beta y_2(t)$$

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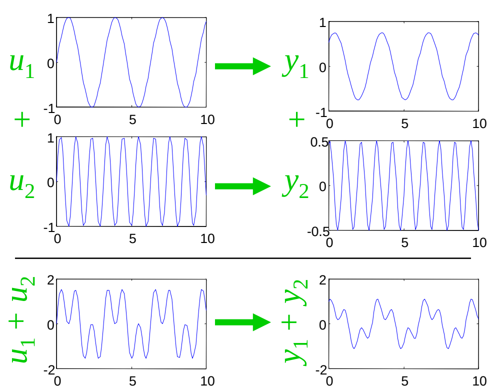
Linear Systems

$u \to$
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$
$\to y$



Input/output linearity at x(0) = 0

- Linear systems are linear in initial condition *and* input ⇒ need to use x(0) = 0 to add outputs together
- For different initial conditions, you need to be more careful (sounds like a good midterm question)

Linear system ⇒ step response and frequency response scale with input amplitude

- 2X input ⇒ 2X output
- Allows us to use *ratios* and *percen-tages* in step/freq response. These are *independent* of input amplitude
- Limitation: input saturation ⇒ only holds up to certain input amplitude

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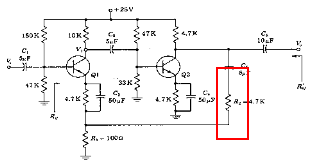
Why are Linear Systems Important?

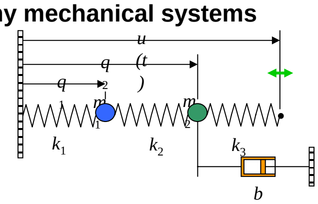
Many important *examples*

Electronic circuits



- Especially true after feedback
- Frequency response is key performance specification (think telephones)

Many mechanical systems



Quantum mechanics, Markov chains, …

Many important *tools*

Frequency response, step response, etc

- Traditional tools of control theory
- Developed in 1930's at Bell Labs; intercontinental telecom

Classical control design toolbox (CDS 101/110a)

- Nyquist plots, gain/phase margin
- Loop shaping

Optimal control and estimators (CDS 110b)

- Linear quadratic regulators
- Kalman estimators

Robust control design (CDS 110b/212)

- H_∞ control design
- μ analysis for structured uncertainty

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Solutions of Linear Systems: The Matrix Exponential

$$\dot{x} = Ax + Bu \qquad y = Cx + Du \longrightarrow y(t) = ???$$

Scalar linear system, with no input

$$\dot{x} = ax \qquad y = cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$

Matrix version, with no input

$$\dot{x} = Ax \qquad y = Cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow \boxed{y(t) = Ce^{At}x_0}$$

initial(A,B,C,D,x0);

Matrix exponential

- Analog to the scalar case; defined by series expansion:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

P = expm(M)

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Stability of Linear Systems

$$\dot{x} = Ax + \not{B}u \;\; (\to 0)$$
$$\not{y = Cx + Du}$$

$$x(t) = e^{At} x_0$$

Q: when is the system asymptotically stable?

$$\lim_{t \to \infty} x(t) = 0$$

Stability is determined by the *eigenvalues* of the matrix *A*

- Simple case: diagonal system

$$\dot{x} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} x \quad \Rightarrow \quad x(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} x_0$$

Stable if $\lambda_i \le 0$
Asy stable if $\lambda_i < 0$
Unstable if $\lambda_i > 0$

- More generally: transform to "Jordan" form

$$\dot{x} = T^{-1} J T x \qquad J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix}$$

Asy stable if $\mathrm{Re}(\lambda_i) < 0$
Unstable if $\mathrm{Re}(\lambda_i) > 0$
Indeterminate if $\mathrm{Re}(\lambda_i) = 0$

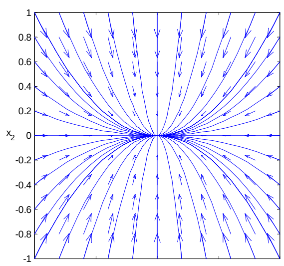
Form of eigenvalues determines system behavior

Linear systems are automatically *globally* stable or unstable

Eigenstructure of Linear Systems

Real e-values

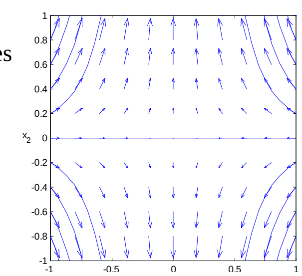
$\mathrm{Re}(\lambda_i) < 0$



Real e-values

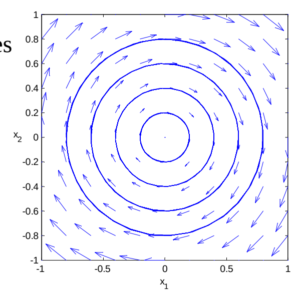
$\mathrm{Re}(\lambda_i) < 0$

$\mathrm{Re}(\lambda_j) > 0$



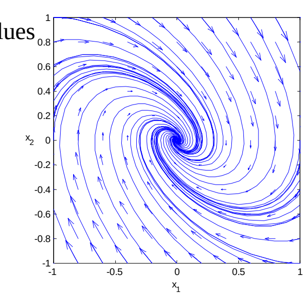
Complex e-values

$\mathrm{Re}(\lambda_i) = 0$



Complex e-values

$\mathrm{Re}(\lambda_i) < 0$



Step and Frequency Response

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

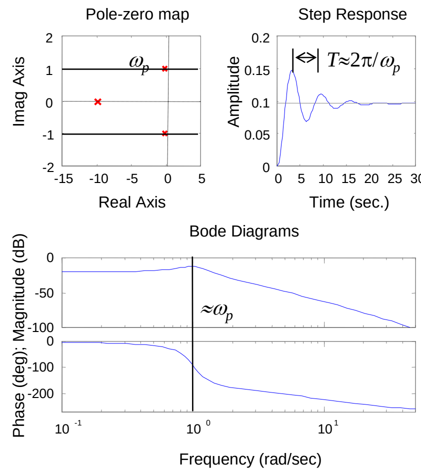
$$u(t) = 1(t) \qquad u(t) = A\sin(\omega t)$$

Effect of eigenstructure on step response

- Complex eigenvalues with small real part lead to oscillatory response
- Frequency of oscillations $\approx \omega_i$

Effects of eigenstructure on frequency response

- Eigenvalues determine “break points” for frequency response
- Complex eigenvalues lead to peaks in response function near ω_i



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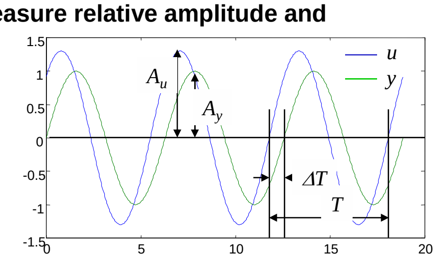
Computing Frequency Responses

Technique #1: plot input and output, measure relative amplitude and phase

- Use MATLAB or SIMULINK to generate response of system to sinusoidal output
- Gain = A_y/A_u
- Phase = $2\pi \cdot \Delta T/T$
- Note: In general, gain and phase will depend on the input amplitude



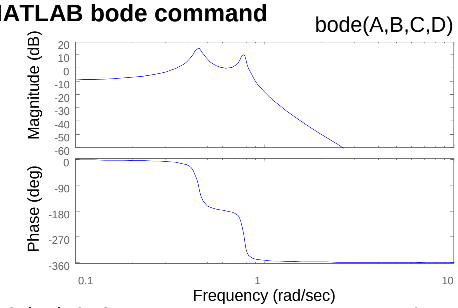
Technique #2 (linear systems): use MATLAB bode command

bode(A,B,C,D)

- Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Gain plotted on log-log scale
  - dB = 20 $\log_{10}$ (gain)
- Phase plotted on linear-log scale



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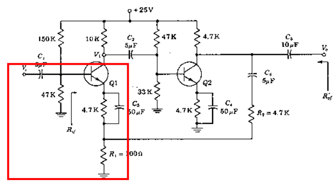
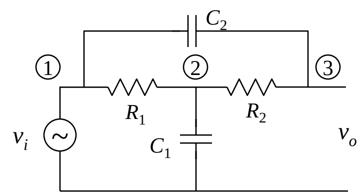
Example: Electrical Circuit





"Bridged Tee Circuit"

Derivation based on Kirchoff's laws for electrical circuits (Ph 2)

- Sum of currents at nodes = 0:

$$C_1 \frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1} - \frac{v_2 - v_3}{R_2} \qquad C_2 \frac{d(v_3 - v_1)}{dt} = -\frac{v_3 - v_2}{R_2}$$

- Rewrite in terms of new states: $v_{c1}=v_2$, $v_{c2}=v_3 - v_1$

$$\frac{d}{dt}\begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \\ V_{c2} \end{bmatrix} v_i \qquad v_o = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + v_i$$

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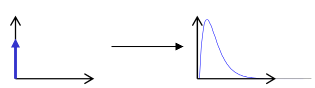
Linear Control Systems and Convolution

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

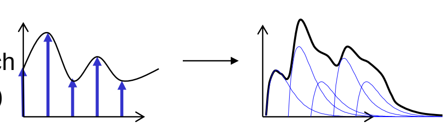
Impulse response, h(t) = Ce^{At}B

- Response to input "impulse"
- Equivalent to "Green's function"



Linearity ⇒ compose response to arbitrary *u*(*t*) using *convolution*

- Decompose input into "sum" of shifted impulse functions
- Compute impulse response for each
- "Sum" impulse response to find *y*(*t*)



Complete solution: use integral instead of "sum"

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input ⇒ 2X output when $x(0) = 0$

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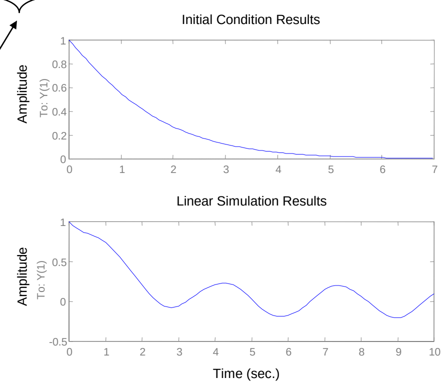
Matlab Tools for Linear Systems

$$y(t) = \underbrace{Ce^{At}x(0)} + \int_{\tau=0}^{t} \underbrace{Ce^{A(t-\tau)}Bu(\tau)d\tau} + Du(t)$$

```
A = [-1 1; 0 -1];  B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```



Other MATLAB commands

- gensig, square, sawtooth – produce signals of diff. types
- step, impulse, initial, lsim – time domain analysis
- bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

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Linearization Around an Equilibrium Point

$$\dot{x} = f(x,u) \qquad \dot{z} = Az + Bv$$
$$y = h(x,u) \qquad w = Cz + Dv$$

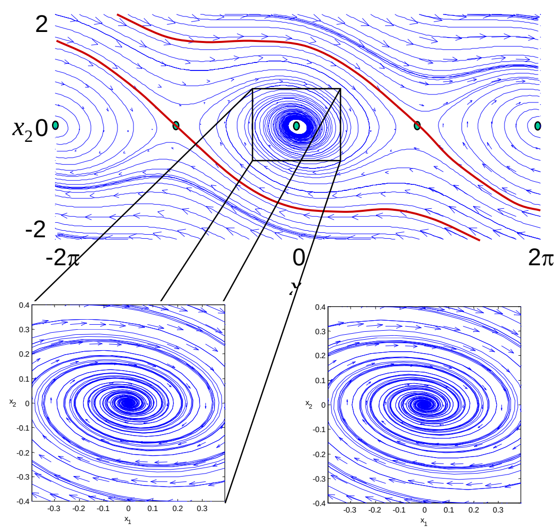
"Linearize" around $x=x_e$

$$f(x_e,u_e) = 0 \quad y_e = h(x_e,u_e)$$
$$z = x - x_e \quad v = u - u_e \quad w = y - y_e$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$
$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point



Full nonlinear model Linear model (honest!)

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Local Stability of Nonlinear Systems

Asymptotic stability of the linearization implies *local* asymptotic stability of equilibrium point

- Linearization around equilibrium point captures “tangent” dynamics

$$\dot{x} = f(x) = A\cdot(x - x_e) + o(x - x_e)$$ ← higher order terms

- If linearization is *unstable,* can conclude that nonlinear system is locally unstable
- If linearization is *stable* but not *asymptotically stable,* can't conclude anything about nonlinear system:

$$\dot{x} = \pm x^3 \xrightarrow{\text{linearize}} \dot{x} = 0$$

- linearization is stable (but not asy stable)
- nonlinear system can be asy stable or unstable

Local approximation particularly appropriate for control systems design

- Control often used to *ensure* system stays near desired equilibrium point
- If dynamics are well-approximated by linearization near equilibrium point, can use this to design the controller that keeps you there (!)

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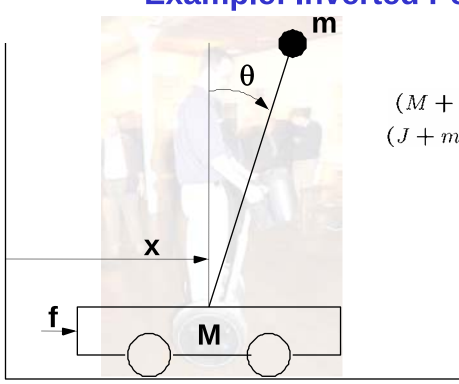
Example: Inverted Pendulum on a Cart



$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around $\theta = \pi$

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{J(M+m)+Mml^2} & \dfrac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \dfrac{mgl(M+m)}{J(M+m)+Mml^2} & \dfrac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{J(M+m)+Mml^2} \\ \dfrac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

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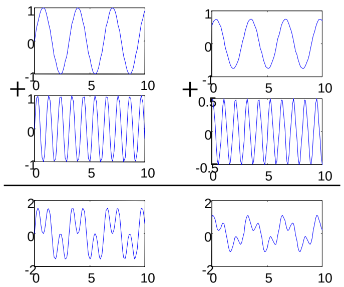
Summary: Linear Systems

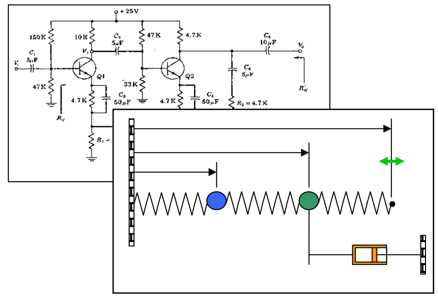
u →

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

→ y



$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$



Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems

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